Approximate Zero-Variance Importance Sampling for Static Network Reliability Estimation

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***Abstract*—We propose a new Monte Carlo method, based on dynamic importance sampling, to estimate the probability that a given set of nodes is connected in a graph (or network) where each link is failed with a given probability. The method generates the link states one by one, using a sampling strategy that approximates an ideal zero-variance importance sampling scheme. The approximation is based on minimal cuts in subgraphs. In an asymptotic rare-event regime where failure probability becomes very small, we prove that the relative error of our estimator remains bounded, and even converges to 0 under additional conditions, when the unreliability of individual links converges to 0. The empirical performance of the new sampling scheme is illustrated by examples.**

***Index Terms*—Monte Carlo methods, network reliability, variance reduction.**

Acronyms

BRE	bounded relative error
CLT	central-limit theorem
IS	importance sampling
MC	Monte Carlo
VRE	vanishing relative error

Notation

$\mathcal{N}$	set of nodes
$\mathcal{L}$	set of links $\{1, \ldots, m\}$
m	number of links
$\mathcal{G}$	undirected graph $(\mathcal{N}, \mathcal{L})$
$\mathcal{K}$	set of target (destination) nodes
γ	a $\mathcal{K}$-cut of $\mathcal{G}$
$E(\gamma)$	event "all links in γ are failed"
$\mathcal{C}$	set of all mincuts of $\mathcal{G}$
q_i	unreliability of link i
u	graph unreliability
X_i	(random) state of link i: 1 if operational, 0 otherwise
X	$(X_1, \ldots, X_m)$: (random) configuration of the graph
x	$= (x_1, \ldots, x_m)$: one realization of X
$\psi(X)$	structure function associated with the $\mathcal{K}$-terminal *unreliability*: $\psi(X) = 1$ if the nodes in $\mathcal{K}$ are *not* connected, 0 otherwise
ϵ	rarity parameter
n	number of s-independent replications
$X^{(j)}$	j-th s-independent replication of X
$\mathbb{P}$	original probability law of the network
$\mathbb{E}$	expectation under $\mathbb{P}$
$\tilde{q}_i$	IS unreliability of link i
$\tilde{\mathbb{P}}$	IS probability law
$\tilde{\mathbb{E}}$	expectation under $\tilde{\mathbb{P}}$
$\mathcal{C}_i$	set of mincuts that contain no link $j \leq i$ for which $x_j = 1$, when $(x_1, \ldots, x_{i-1})$ are fixed
γ_i	a mincut of maximal probability in $\mathcal{C}_i$.
$u_i(\cdot)$	$u_i(x_1, \ldots, x_{i-1})$ is the unreliability of the graph when $(x_1, \ldots, x_{i-1})$ are fixed
$\widehat{u}_i(\cdot)$	$\widehat{u}_i(x_1, \ldots, x_{i-1})$ is an approximation of $u_i(x_1, \ldots, x_{i-1})$
$L_i(x_i)$	$= x_i(1-q_i)/(1-\widetilde{q}_i) + (1-x_i)q_i/\widetilde{q}_i$: likelihood ratio associated with step i
$L(x)$	$\prod_{i=1}^{m} L_i(x_i)$: global likelihood ratio
o	For two functions $f, g : (0, \infty) \to \mathbb{R}$, we say that $f(\epsilon) = o(g(\epsilon))$ if $\lim_{\epsilon \to 0^+} f(\epsilon)/g(\epsilon) = 0$
O	$f(\epsilon) = O(g(\epsilon))$ if $\lvert f(\epsilon)\rvert \leq c_1 g(\epsilon)$ for some constant $c_1 > 0$ for all ϵ sufficiently small
$\underline{O}$	$f(\epsilon) = \underline{O}(g(\epsilon))$ if $\lvert f(\epsilon)\rvert \geq c_2 g(\epsilon)$ for some constant $c_2 > 0$ for all ϵ sufficiently small
Θ	$f(\epsilon) = \Theta(g(\epsilon))$ if $f(\epsilon) = \underline{O}(\epsilon^d)$, and $f(\epsilon) = O(\epsilon^d)$

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Definitions & Nomenclature

- A *cut* (or $\mathcal{K}$-cut) in the graph $\mathcal{G}$ is a set of edges such that, if we remove them from $\mathcal{G}$, then the nodes in $\mathcal{K}$ are not in the same connected component of the resulting graph.
- A *mincut* (minimal cut) is a cut that contains no other cut than itself.

I. INTRODUCTION

THE STATIC network reliability problem, which consists in computing the probability that a given set of nodes in a graph is connected when each individual link is failed with a given probability, occurs in a wide range of applications [4], [7], [11]. Examples are easily found in telecommunications, where we may want to assess the probability that a selected group of nodes (which can be just a pair) can communicate; or in power supply systems, where we may want to estimate the risk that electricity is not provided to certain nodes; or in transportation systems, where links represent the roads and are subject to damages. In all these settings, a disconnected set of nodes may have critical implications, either financially or security-wise, and an accurate reliability estimate is needed. For large graphs, exact computation of the *unreliability* (the probability of *system failure*, which happens when the considered set of nodes is not connected) is impractical because the problem is NP-hard in general; for all known algorithms, the required computational effort increases exponentially with the number of links [3], [10], [25].

Monte Carlo methods can estimate unreliability for very large graphs. In its crude form, the Monte Carlo algorithm samples n s-independent realizations of the graph, and estimates the unreliability u by the proportion of these n realizations for which the selected nodes are not connected. However, this simple approach becomes useless when the network is highly reliable (u is very close to 0), because one would need an excessively large value of n to obtain enough realizations where the nodes are not connected. For example, if $u = 10^{-10}$, we expect one such realization per ten billion runs on average, and we need many more than that to estimate u even with only a single digit of accuracy. Such small values of u are commonplace in critical applications.

Special variance reduction techniques have been developed to address this type of *rare-event* estimation problem. Some of them are based on the *importance sampling* (IS) principle, which consists in changing the sampling probabilities of the links so that system failure is no longer a rare event, and multiplying the original estimator by an appropriate likelihood ratio to recover unbiasedness [2], [16], [19], [27]. Specific IS methods have been proposed for estimating the unreliability in static graphs; see [15], [24], or [7], and the references given there. The main difficulty with IS is to find an effective way of changing the sampling probabilities. If the probabilities of realizations that lead to failure are inflated unevenly, or if some of them are decreased too much, then the likelihood ratio may have a huge variance, and this may result in a badly behaved estimator even if system failure is no longer a rare event [19]. This pitfall is hard to avoid in general, and is often amplified when the failure probabilities approach 0.

Robustness of estimators in this type of situation is usually studied by examining the behavior of the relative error (the standard deviation divided by the mean) as a function of rarity [12], [13], [16], [18], [19]. Two characterizations examined in this paper are the widely-used *bounded relative error* (BRE) property, which holds when the relative error remains bounded regardless of the rarity [13], [16]; and the stronger *vanishing relative error* (VRE) property, which says that the relative error converges to zero when the unreliability u goes to zero [18]. These properties are particularly relevant in situations where u is very small: the relative width of a confidence interval on u based on the central-limit theorem (CLT) for a fixed sample size n when $u \to 0$ remains bounded if BRE holds, and converges to zero if VRE holds [18].

The aim of this paper is to propose and study a new way of applying IS to address the static network reliability problem. The proposed IS strategy is based on a (theoretical) zero-variance IS scheme, which exists in principle whenever the original estimator cannot take negative values [2], [16], [19], [20]. We represent the sampling process of all links of the graph by a Markov chain that determines one new link at each step, and whose state is the states of all links that have already been sampled. We show how the transition probabilities of this Markov chain can be changed (at least in principle) to obtain a zero-variance IS estimator of u. Under this ideal IS scheme, the system always fails, and the likelihood ratio is always equal to u, so the IS estimator is a constant, with probability 1. Unfortunately, implementing this scheme directly is not practically possible unless we can compute everything exactly in the first place. But it can be approximated, and this is what we do here.

We propose an approximation based on minimal cuts having (relatively) high failure probability in the subgraph that remains after removing the links already known to be failed, and enforcing the states of the links known to be operational, at each step of the Markov chain. These cuts are used to approximate the unreliability conditional on the current state, at each step. We prove that our estimators enjoy the BRE property, and that under an additional condition on the approximation of the conditional unreliability at each step, the VRE property is also verified. This type of asymptotic analysis as a function of a rarity parameter has been done extensively for queuing systems [5], [13] and for highly reliable Markovian systems [16], [21], [22], [26], [30]. For static network reliability, as far as we know, the only proposed estimator with the BRE property is the path-based estimator described in [8], [9], which was shown to have this property but under more restrictive conditions than those made here, and no estimator with the VRE property has been proposed so far. Our numerical results on various examples show that the method introduced here performs quite well even on large graphs with extremely small unreliabilities, and the VRE property can be observed empirically.

The paper is organized as follows. Section II presents the mathematical model. In Section III, we recall the *crude* Monte Carlo method, we show its inefficiency, and we explain how IS can be applied in general for this model. In Section IV, we construct an ideal (zero-variance) IS estimator. Its implementation would require the knowledge of the conditional unreliability (function) given the status of arbitrary subsets of links, which is of course utopian; but an estimate of this conditional unreliability can be plugged into the formula in place of the exact one. We prove that this yields an estimator with BRE under appropriate conditions on the conditional reliability estimate, and with VRE under an additional condition. In Section V, we propose a specific (crude) approximation method for this conditional unreliability, based on the probability of a minimal cut. We prove that the resulting IS estimator always has the BRE property, and the VRE property under additional conditions that are often verified in our examples. In Section VI, we illustrate numerically the efficiency of the method on several examples. Section VII provides a conclusion, and discusses issues worthy of future work.

II. Model

We consider an undirected connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ where $\mathcal{N}$ is the set of nodes, and $\mathcal{L} = \{1, \ldots, m\}$ is the set of links. We focus here on a static model, where time plays no explicit role. Nodes never fail, but links are subject to s-independent failures, link $i \in \mathcal{L}$ failing with probability q_i, where $0 < q_i < 1$. A *configuration* [25] of the graph is given by the random vector

$$X = (X_1, \ldots, X_m)$$

where for all $i \in \mathcal{L}$, $X_i = 1$ if link i works, and $X_i = 0$ if link i is failed. By retaining only the set of operational links $\mathcal{L}' \subseteq \mathcal{L}$, we end up with a (random) partial graph $\mathcal{G}' = (\mathcal{N}, \mathcal{L}')$ of $\mathcal{G}$. Knowing the vector X is equivalent to knowing $\mathcal{G}'$.

Our goal is to estimate the probability u that a given set of nodes $\mathcal{K}$ are not connected in the random graph $\mathcal{G}'$, or equivalently for the random configuration X. Formally, we define $\psi(X)$ by $\psi(X) = 1$ if the set of nodes $\mathcal{K}$ is *not* connected in $\mathcal{G}'$, that is, they do not belong to the same connected component when the configuration is X, and $\psi(X) = 0$ otherwise. The expectation $u = \mathbb{E}[\psi(X)]$ is the *$\mathcal{K}$-terminal unreliability*. Note that, in the reliability literature, it is more customary to work with the function ϕ defined by $\phi(X) = 1 - \psi(X)$, so that the network reliability is $r = \mathbb{E}[\phi(X)]$. We use ψ instead of ϕ because it simplifies our notation significantly, and because it is more in line with the usual setting of rare event simulation, where an indicator function is equal to 1 when the rare event occurs and the goal is to estimate its expectation. The most frequent case is when estimating the *two-terminal* or *source-to-terminal unreliability*, where $\mathcal{K}$ is comprised of only two nodes.

This unreliability metric can be written as

$$\begin{aligned} u &= \mathbb{E}[\psi(X)] = \sum_{x\in\{0,1\}^m} \psi(x)\mathbb{P}[X = x] \\ &= \sum_{x\in\{0,1\}^m} \psi(x) \prod_{i=1}^{m} (q_i(1 - x_i) + (1 - q_i)x_i) \end{aligned}$$

where $x = (x_1, \ldots, x_m)$. The state space (the set of all possible configurations) is of cardinality 2^m. This means that computing u directly from this formula requires a time that increases exponentially with m, the number of links. As we said earlier, this problem is NP-hard in general [3], so for large graphs, approximation techniques are required, and Monte Carlo simulation becomes the method of choice when the graph is large enough.

We are interested mainly in the situation where the failure probabilities q_i of the individual links are very close to 0. We will capture this by studying the performance of our algorithm in an asymptotic regime where the q_i are polynomial functions of a rarity parameter $\epsilon \ll 1$, as done for example in [18], [23], [31]. More specifically, we shall assume that, for each $i \in \mathcal{L}$, there are constants $a_i > 0$, and $b_i \geq 0$ (independent of ϵ) such that

$$q_i = a_i \epsilon^{b_i}. \tag{1}$$

Under this assumption, it is easy to verify (because the configuration space is finite, and the probability of any configuration is a polynomial in ϵ) that the system unreliability (which depends on ϵ) is

$$u = u(\epsilon) = \Theta(\epsilon^c) \tag{2}$$

for some constant $c \geq 0$ (for similar remarks, see [10] or [25], for instance). We further assume that $c > 0$, so that $u(\epsilon) \to 0$ when $\epsilon \to 0$. A sufficient condition for this result is that $b_i > 0$ for all i. A necessary and sufficient condition is given in Section V-B.

III. Monte Carlo Simulation

The crude *Monte Carlo* (MC) method simulates n s-independent realizations of X, say $X^{(1)}, \ldots, X^{(n)}$, and estimates u by the proportion of those realizations where the nodes in $\mathcal{K}$ are not connected:

$$\bar{U}_{\mathrm{MC}}^{(n)} = \frac{1}{n} \sum_{i=1}^{n} \psi(X^{(i)}).$$

When n is large enough, thanks to the central-limit theorem (CLT), we can in principle compute a confidence interval on u, say at confidence level α, by assuming that $\bar{U}_{\rm MC}^{(n)}$ is approximately normally distributed. This gives the confidence interval

$$\left(\bar{U}_{\rm MC}^{(n)} - c_\alpha S^{(n)}/\sqrt{n},\ \bar{U}_{\rm MC}^{(n)} + c_\alpha S^{(n)}/\sqrt{n}\right) \qquad (3)$$

with confidence $1-\alpha$, where c_α is the $1-\alpha/2$ quantile of the standard normal distribution (with mean 0 and variance 1), and $(S^{(n)})^2 = \bar{U}_{\rm MC}^{(n)}(1-\bar{U}_{\rm MC}^{(n)})n/(n-1)$ is the empirical variance of $\psi(X^{(1)}), \ldots, \psi(X^{(n)})$.

The expected relative half-width of this confidence interval is

$$c_\alpha \frac{(\mathrm{Var}[\psi(X)]/n)^{1/2}}{\mathbb{E}[\psi(X)]} = c_\alpha \left(\frac{1-u}{un}\right)^{1/2},$$

which increases to ∞ when $u \to 0$ for fixed n. To keep it under control, we need $n = \underline{O}(1/u)$. This means that, in the asymptotic regime defined by (1), we would need $n = n(\epsilon) = \underline{O}(\epsilon^{-c})$.

Our aim is to replace $\psi(X)$ by an alternative unbiased estimator Y with smaller variance. Numerous techniques have been designed to do that; we refer the reader to [7] for an overview. For fixed n, and $\psi(X)$ replaced by Y, the half-width of the confidence interval above is proportional to the *relative error* of Y, defined as $\mathrm{RE}[Y] = (\mathrm{Var}[Y])^{1/2}/\mathbb{E}[Y]$. For $Y = \psi(X)$, the relative error is $\Theta(u^{-1/2})$. We are interested in having an estimator Y with *bounded relative error* (BRE), which means that $\mathrm{RE}[Y]$ remains bounded when $u \to 0$, or in our framework when $\epsilon \to 0$. It would be even better if Y has the stronger property of *vanishing relative error* (VRE), which means that $\mathrm{RE}[Y] \to 0$ when $\epsilon \to 0$. To our knowledge, no VRE estimator has been constructed so far for this problem.

IV. Importance Sampling and Zero-Variance Approximation

A. Importance Sampling

IS consists in replacing the probabilities of the 2^m possible configurations X by another set of probabilities [2], [16], [19]. If we denote the original probabilities by $\mathbb{P}$, and the new ones by $\widetilde{\mathbb{P}}$, we have

$$\begin{aligned} u = \mathbb{E}[\psi(X)] &= \sum_{x\in\{0,1\}^m} \psi(x)\mathbb{P}[X=x] \\ &= \sum_{x\in\{0,1\}^m} \psi(x)L(x)\widetilde{\mathbb{P}}[X=x] \end{aligned}$$

where

$$L(x) = \mathbb{P}[X=x]/\widetilde{\mathbb{P}}[X=x],$$

provided that $\widetilde{\mathbb{P}}[X=x] > 0$ whenever $\psi(x)\mathbb{P}[X=x] > 0$. This $L(x)$ is the *likelihood ratio* of the old and new probabilities for outcome x. The unreliability can then be rewritten as

$$u = \widetilde{\mathbb{E}}[\psi(X)L(X)] \qquad (4)$$

where $\widetilde{\mathbb{E}}[\cdot]$ is the expectation under $\widetilde{\mathbb{P}}$. We thus have the unbiased IS estimator of u:

$$\bar{U}_{\rm IS}^{(n)} = \frac{1}{n}\sum_{j=1}^{n} \psi(X^{(j)})L(X^{(j)}),$$

where $X^{(1)}, \ldots, X^{(n)}$ are s-independent copies of X all distributed according to $\widetilde{\mathbb{P}}$. A confidence interval on u can be computed as in (3), but with $\bar{U}_{\rm MC}^{(n)}$, and $(S^{(n)})^2$ replaced by the sample mean, and the sample variance of the $\psi(X^{(j)})L(X^{(j)})$.

The optimal change of probabilities in this setting inflates all the probabilities by a factor proportional to $\psi(x)$ [2], [19]. This result gives

$$\widetilde{\mathbb{P}}[X=x] = \psi(x)\mathbb{P}[X=x]/u$$

for all $x \in \{0,1\}^m$. That is, the realizations where the system does not fail are given zero probability, and the probabilities of the other realizations are rescaled accordingly (divided by u). Under these new probabilities, with probability 1, the system fails, and the likelihood ratio is u, so the estimator always takes the value u. Thus, this estimator has zero variance. However, implementing this IS scheme requires knowing u in the first place.

B. A Sequential Version of Zero-Variance Importance Sampling

We now reformulate the sampling of X as a Markov chain process, and we define a zero-variance IS scheme for that process by adapting the techniques described in [17], [19], [20]. The link states $X_1, \ldots, X_m$ are generated successively in this order. At step i of the chain, we generate the coordinate X_i of X, over the set $\{0,1\}$. Under $\mathbb{P}$, these probabilities are $\mathbb{P}[X_i = 0] = 1 - \mathbb{P}[X_i = 1] = q_i$, and the X_i are s-independent. Under IS, we will change the probability q_i at each step, and let it depend on the previously generated values $X_1, \ldots, X_{i-1}$. We will see that, by doing this optimally, one can obtain a zero-variance estimator.

We define

$$u_i(x_1, \cdots, x_{i-1}) = \mathbb{E}[\psi(X) | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}],$$

which represents the unreliability of graph $\mathcal{G}'$ conditional on the states of links 1 to $i-1$, which are already determined before step i. The unconditional unreliability of the graph can be written as $u = u_1(\emptyset)$. We will often use the fact that

$$u_i(x_1, \cdots, x_{i-1}) = q_i u_{i+1}(x_1, \ldots, x_{i-1}, 0) + (1 - q_i) u_{i+1}(x_1, \ldots, x_{i-1}, 1). \quad (5)$$

Suppose that, for $i = 1, \ldots, m$, we replace q_i by

$$\begin{aligned} \widetilde{q}_i &\stackrel{\text{def}}{=} \widetilde{\mathbb{P}}[X_i = 0 \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \\ &= q_i \frac{u_{i+1}(x_1, \ldots, x_{i-1}, 0)}{u_i(x_1, \ldots, x_{i-1})}. \end{aligned} \quad (6)$$

This result gives

$$1 - \widetilde{q}_i = (1 - q_i) \frac{u_{i+1}(x_1, \ldots, x_{i-1}, 1)}{u_i(x_1, \ldots, x_{i-1})}. \quad (7)$$

Using (6), and (7), we find that, for a given realization $x = (x_1, \ldots, x_m)$, the likelihood ratio for step i is

$$\begin{aligned} L_i(x_i) &= x_i \frac{1 - q_i}{1 - \widetilde{q}_i} + (1 - x_i) \frac{q_i}{\widetilde{q}_i} \\ &= \frac{u_i(x_1, \ldots, x_{i-1})}{u_{i+1}(x_1, \ldots, x_{i-1}, x_i)}. \end{aligned}$$

The overall likelihood ratio is

$$L(x) = \prod_{i=1}^{m} L_i(x_i)$$

and the final estimator is $\psi(X)L(X)$.

The following theorem says that this sequential (state-dependent) IS scheme gives a zero-variance estimator. It is a consequence of the general results on zero-variance sampling for Markov chains, given in [17], [19], [20]. We give a direct proof from first principles.

Theorem 1: Under the sequential IS algorithm where the probabilities q_i are replaced by the conditional probabilities $\widetilde{q}_i$ defined in (6), with probability 1, we have $\psi(X) = 1$, and $L(X) = u$, so the IS estimator has zero variance.

Proof: We first show that $\widetilde{\mathbb{P}}[\psi(X) = 1] = 1$. Let x be a configuration for which the system works, that is, $\psi(x) = 0$; and suppose that k is the index of the first coordinate such that $\psi(x_1, \ldots, x_k, \ldots) = 0$ whatever be the values of $(x_{k+1}, \ldots, x_m)$, for the given (fixed) values of $x_1, \ldots, x_k$. In this case, $u_{k+1}(x_1, \ldots, x_k) = 0$, which implies from (6) that conditional on $x_1, \ldots, x_{k-1}$, this particular x_k has zero probability. Therefore, every configuration x with $\psi(x) = 0$ has zero probability. We also have

$$\begin{aligned} L(x) &= \prod_{i=1}^{m} L_i(x_i) = \prod_{i=1}^{m} \frac{u_i(x_1, \ldots, x_{i-1})}{u_{i+1}(x_1, \ldots, x_{i-1}, x_i)} \\ &= \frac{u_1(\emptyset)}{u_{m+1}(x_1, \ldots, x_m)} = \frac{u}{u_{m+1}(x_1, \ldots, x_m)}. \end{aligned}$$

For any configuration x with $\psi(x) = 1$, we have $u_{m+1}(x_1, \ldots, x_m) = \psi(x_1, \ldots, x_m) = 1$, and therefore $L(x) = u$. This result means that, with probability 1, we have $\psi(X) = 1$, and $\psi(X)L(X) = u$. ■

Algorithm 1 below summarizes this zero-variance IS scheme.

Algorithm 1 Exact Zero-Variance Simulation

```
L ← 1;
for i = 1 to m do
    q̃_i ← q_i u_{i+1}(x_1, ..., x_{i-1}, 0)/u_i(x_1, ..., x_{i-1});
    generate U_i a uniform random variate over (0, 1);
    if U_i < q̃_i then
        x_i ← 0;      L_i ← q_i/q̃_i;
    else
        x_i ← 1;      L_i ← (1 − q_i)/(1 − q̃_i);
    end if
    L ← L × L_i;
end for
return Y = L as the exact unreliability;
```

Implementing this zero-variance IS scheme would require the perfect knowledge of all the functions u_i, and in particular the knowledge of $u_1(\emptyset) = u$, so it is not practical. In the next subsection, we study the situation where each u_i is replaced by some approximation $\widehat{u}_i$. One way to obtain such an approximation will be proposed in Section V-A.

C. Zero-Variance Approximation, and Robustness Properties

The idea of zero-variance approximation is to replace the functions $u_i(\cdot)$ in (6) by some easy-to-compute approximations $\widehat{u}_i(\cdot)$. This gives

$$\begin{aligned} \widetilde{q}_i &= \widetilde{\mathbb{P}}[X_i = 0] \\ &= \frac{q_i \widehat{u}_{i+1}(x_1, \ldots, x_{i-1}, 0)}{q_i \widehat{u}_{i+1}(x_1, \ldots, x_{i-1}, 0) + (1 - q_i) \widehat{u}_{i+1}(x_1, \ldots, x_{i-1}, 1)}. \end{aligned} \quad (8)$$

The intuition is that, if $\widehat{u}_{i+1}(\cdot)$ is not too far from $u_{i+1}(\cdot)$ for each i, then the sampling will be done with probabilities that are not too far from the optimal ones, and the variance can then be reduced by large factors. Observe that the network unreliability u will not change if we modify the order in which the edges are enumerated and visited in the graph, but the change of measure in our proposed approximate algorithm does depend on that ordering in general. Unfortunately, we do not have an efficient way of optimizing the ordering, nor a good robust heuristic to choose it. However, in our experiments with the mincut-maxprob approximation of Section V-A, just taking a random or arbitrary ordering has worked fine in all the examples we tried.

The next two theorems characterize that intuition in an asymptotic regime where $\epsilon \to 0$, while the graph remains fixed. They give sufficient conditions on the approximations $\widehat{u}_i(\cdot)$ for the BRE or VRE property to hold.

Theorem 2: Suppose that for each i and $(x_1, \dots, x_i) \in \{0,1\}^i$, $1 \le i \le m$, there is a constant $a_{i+1}(x_1, \dots, x_i)$ independent of ϵ such that

$$\begin{aligned}\widehat{u}_{i+1}(x_1, \dots, x_i) &= a_{i+1}(x_1, \dots, x_i) u_{i+1}(x_1, \dots, x_i) \\ &\quad + o(u_{i+1}(x_1, \dots, x_i)). \qquad (9)\end{aligned}$$

Then the estimator provided by the IS scheme with change of probabilities defined by (8) has the BRE property.

Proof: We need to show that, under the IS scheme, $\mathrm{Var}[\psi(X)L(X)] = O(u^2)$, or equivalently that $\widetilde{\mathbb{E}}[\psi(X)L^2(X)] = O(u^2)$. From (5), we see that

$$(1-q_i)u_{i+1}(x_1, \cdots, x_{i-1}, 1) = O(u_i(x_1, \cdots, x_{i-1})),$$

and

$$q_i u_{i+1}(x_1, \cdots, x_{i-1}, 0) = O(u_i(x_1, \cdots, x_{i-1})),$$

with at least one of these two terms being $\Theta(u_i(x_1, \cdots, x_{i-1}))$. Using this result, and given that $a_{i+1}(x_1, \dots, 1)$ and $a_{i+1}(x_1, \dots, 0)$ are constants independent of ϵ, one can verify that there is a positive constant $b_i(x_1, \dots, x_{i-1})$ independent of ϵ, and such that

$$\begin{aligned}&(1-q_i)\widehat{u}_{i+1}(x_1, \cdots, x_{i-1}, 1) + q_i \widehat{u}_{i+1}(x_1, \cdots, x_{i-1}, 0) \\ &= (1-q_i) a_{i+1}(x_1, \dots, 1) u_{i+1}(x_1, \cdots, x_{i-1}, 1) \\ &\quad + q_i a_{i+1}(x_1, \dots, 0) u_{i+1}(x_1, \cdots, x_{i-1}, 0) \\ &\quad + o(u_{i+1}(x_1, \dots, 1) + u_{i+1}(x_1, \dots, 0)) \\ &= b_i(x_1, \dots, x_{i-1}) u_i(x_1, \cdots, x_{i-1}) + o(u_i(x_1, \cdots, x_{i-1})).\end{aligned}$$

For example, if

$$(1-q_i)u_{i+1}(x_1, \cdots, x_{i-1}, 1) = o(u_i(x_1, \cdots, x_{i-1})),$$

then

$$\begin{aligned}&a_{i+1}(x_1, \dots, 0) u_i(x_1, \cdots, x_{i-1}) \\ &= q_i a_{i+1}(x_1, \dots, 0) u_{i+1}(x_1, \cdots, x_{i-1}, 0) \\ &\quad + o(u_{i+1}(x_1, \dots, x_{i-1}),\end{aligned}$$

so we have $b_i(x_1, \dots, x_{i-1}) = a_{i+1}(x_1, \dots, 0)$. The other cases are similar.

The likelihood ratio for the sampling of link i is then

$$\begin{aligned}L_i(X_i) &= \frac{(1-q_i)\widehat{u}_{i+1}(X_1, \cdots, X_{i-1}, 1)}{\widehat{u}_{i+1}(X_1, \cdots, X_{i-1}, X_i)} \\ &\quad + \frac{q_i \widehat{u}_{i+1}(x_1, \cdots, x_{i-1}, 0)}{\widehat{u}_{i+1}(X_1, \cdots, X_{i-1}, X_i)} \\ &= \frac{b_i(X_1, \dots, X_{i-1})}{a_{i+1}(X_1, \dots, X_i)} \frac{u_i(X_1, \dots, X_{i-1})}{u_{i+1}(X_1, \dots, X_i)} + o(1).\end{aligned}$$

Let

$$d = \max_{\{(x_1, \cdots, x_i) : 1 \le i \le m\}} \frac{b_i(x_1, \dots, x_{i-1})}{a_{i+1}(x_1, \dots, x_i)}.$$

We have $d < \infty$, because this maximum is over a finite set, and the likelihood ratio for the whole sample path satisfies

$$\begin{aligned}L(X) &= \prod_{i=1}^{m} L_i(X_i) \\ &\le \prod_{i=1}^{m} d \frac{u_i(X_1, \dots, X_{i-1})}{u_{i+1}(X_1, \dots, X_{i-1}, X_i)} + o(u) \\ &\le d^m u + o(u).\end{aligned}$$

Therefore,

$$\widetilde{\mathbb{E}}[\psi(X)L^2(X)] \le \widetilde{\mathbb{E}}[L^2(X)] \le d^{2m}u^2 + o(u^2),$$

due to the finite sum in the expectation, which completes the proof. ■

In the next theorem, we additionally require that for all configurations x that are not rare under IS, at each step i of the method, the unreliability estimations lead to a sampling probability asymptotically equivalent to the zero-variance one, and we show that VRE holds under these conditions. We define

$$\mathcal{S}_1 = \{x \in \{0,1\}^m : \psi(x) = 1 \text{ and } \widetilde{\mathbb{P}}[X = x] = \Theta(1)\},$$

and

$$\mathcal{S}_0 = \{x \in \{0,1\}^m : \psi(x) = 1 \text{ and } \widetilde{\mathbb{P}}[X = x] = o(1)\}.$$

The union $\mathcal{S}_0 \cup \mathcal{S}_1$ is the set of configurations where the system fails. The configurations in $\mathcal{S}_1$ are no longer rare under IS, while those in $\mathcal{S}_0$ remain rare under IS. Our additional conditions for VRE will involve only the configurations $x \in \mathcal{S}_1$.

Theorem 3: Let the assumptions of Theorem 2 hold, and suppose also that, for all $x = (x_1, \dots, x_m) \in \mathcal{S}_1$, and for each i, one of the following three conditions is satisfied:

$$\begin{aligned}&\frac{\widehat{u}_{i+1}(x_1, \dots, x_{i-1}, 1)}{u_{i+1}(x_1, \dots, x_{i-1}, 1)} \\ &= \frac{\widehat{u}_{i+1}(x_1, \dots, x_{i-1}, 0)}{u_{i+1}(x_1, \dots, x_{i-1}, 0)} + o(1), \qquad (10)\end{aligned}$$

or $x_i = 0$, $a_{i+1}(x_1, \dots, x_i) = 1$, and

$$(1-q_i)\widehat{u}_{i+1}(x_1, \dots, x_{i-1}, 1) = o(q_i \widehat{u}_{i+1}(x_1, \dots, x_{i-1}, 0)) \qquad (11)$$

or $x_i = 1$, $a_{i+1}(x_1, \dots, x_i) = 1$, and

$$q_i \widehat{u}_{i+1}(x_1, \dots, x_{i-1}, 0) = o((1-q_i)\widehat{u}_{i+1}(x_1, \dots, x_{i-1}, 1)). \qquad (12)$$

Then the estimator from the IS scheme defined via (8) has the VRE property. In particular, if the assumptions of

Theorem 2 hold with $a_{i+1}(x_1,\dots,x_i)=1$ for all $x\in\mathcal{S}_1$, and all i, then Condition (10) always hold.

Proof: We decompose the second moment in two terms:

$$\widetilde{\mathbb{E}}[\psi(X)L^2(X)] = \sum_{x\in\mathcal{S}_1}\psi(x)L^2(x)\widetilde{\mathbb{P}}[X=x] + \sum_{x\in\mathcal{S}_0}\psi(x)L^2(x)\widetilde{\mathbb{P}}[X=x]. \quad (13)$$

For the second term, following the same argument as in the proof of Theorem 2, we find that there exists a constant $d>0$, independent of ϵ, such that $L^2(x)\le d^{2m}u^2+o(u^2)$; and because $\mathcal{S}_0$ is finite,

$$\sum_{x\in\mathcal{S}_0}\psi(x)L^2(x)\widetilde{\mathbb{P}}[X=x] \le (d^{2m}u^2+o(u^2))\sum_{x\in\mathcal{S}_0}\psi(x)\widetilde{\mathbb{P}}[X=x].$$

But this last sum is $o(1)$, because it is a *finite* sum of $o(1)$ terms. Consequently,

$$\sum_{x\in\mathcal{S}_0}\psi(x)L^2(x)\widetilde{\mathbb{P}}[X=x]=o(u^2).$$

We now focus on the first term on the right side of (13). For the special case where the conditions of Theorem 2 hold with $a_{i+1}(x_1,\dots,x_i)=1$ for all $x\in\mathcal{S}_1$, and all i, we can take $b_i(x_1,\dots,x_{i-1})=1$, and $d=1$ in the proof of that theorem. It gives

$$\sum_{x\in\mathcal{S}_1}\psi(x)L^2(x)\widetilde{\mathbb{P}}[X=x]=u^2+o(u^2). \quad (14)$$

For the more general case, when (10) holds, we have $b_i(x_1,\dots,x_{i-1})=a_{i+1}(x_1,\dots,1)=a_{i+1}(x_1,\dots,0)$. When either (11) or (12) holds, we find from (9), and (5) that $b_i(x_1,\dots,x_{i-1})=a_{i+1}(x_1,\dots,x_i)=1$. So in all three cases, we can take $d=1$, and we also obtain (14).

Combining the results for the two terms, we have

$$\widetilde{\mathrm{Var}}[\psi(X)L(X)]\le\widetilde{\mathbb{E}}[L^2(X)]-u^2=o(u^2),$$

and this concludes the proof. ■

The next question is how to construct approximations $\widehat{u}_{i+1}(\cdot)$ for which these theorems apply. We make one proposal in the next section.

V. An Approximation Based on Minimal Cuts

We introduce a method to approximate $\widehat{u}_{i+1}(\cdot)$ based on the probabilities of certain minimal cuts in subgraphs obtained at each step. We call this method the *mincut-maxprob* approximation. This approximation has the advantage of being relatively simple, and easy to compute. It is not the only possibility; other approximations based on other types of cuts or on combinations of cuts might also work well. It is important to emphasize that the proposed approximation does not involve the rarity parameter ϵ used for the parametric asymptotic analysis, but only the numerical values of the q_i. In Section V-B, we analyze the asymptotic properties of our approximation in terms of ϵ. We show that it always satisfies the condition of Theorem 2, so it always provides an estimator with BRE, and that under additional conditions the estimator also has VRE.

A. The Proposed Approximation and Algorithm

Let $\mathcal{C}$ be the set of all mincuts of $\mathcal{G}$. With any cut γ, we associate the event $E(\gamma)$: "all links in γ are failed." Observe that the system unreliability u can be expressed as

$$u=\mathbb{P}\left[\cup_{\gamma\in\mathcal{C}}E(\gamma)\right], \quad (15)$$

because the system is non-operational iff at least one of its mincuts has all its links failed. A *mincut with maximal probability* is a mincut γ such that

$$\gamma=\arg\max_{\gamma'\in\mathcal{C}}\mathbb{P}[E(\gamma')].$$

For each i, given $x_1,\dots,x_i$ (assumed fixed), consider the graph $\mathcal{G}_i=\mathcal{G}_i(x_1,\dots,x_i)$ obtained from $\mathcal{G}$ by removing all links $j\le i$ for which $x_j=0$, and forcing the links j such that $x_j=1$ to be operational. Let $\mathcal{C}_i$ be the set of mincuts in $\mathcal{G}_i$ that contain no link $j\le i$ for which $x_j=1$. Let γ_i be a mincut of maximal probability in $\mathcal{C}_i$. That is,

$$p_i\stackrel{\text{def}}{=}\mathbb{P}[E(\gamma_i)]=\max\left\{\mathbb{P}[E(\gamma)]:\gamma\in\mathcal{C}_i\right\}.$$

When $E(\gamma_i)$ occurs, all links in γ_i are failed so $\mathcal{K}$ is necessarily disconnected, and γ_i is a cut of largest probability having this property. We disallow γ_i to contain links $j\le i$ with $x_j=1$ because we already know that these links are operational. We will use p_i as our approximation $\widehat{u}_{i+1}(x_1,\dots,x_i)$ of $u_{i+1}(x_1,\dots,x_i)$ at step i. In our asymptotic analysis, when the model is parameterized by ϵ, in the next subsection, we will see that p_i and $u_{i+1}(x_1,\dots,x_i)$ are of the same order in ϵ, and this has motivated our choice of p_i. Intuitively, this is where the BRE property of our method (proved in Theorem 4) comes from.

At step i of the Markov chain, we need $\widehat{u}_{i+1}(x_1,\dots,x_{i-1},0)$ and $\widehat{u}_{i+1}(x_1,\cdots,x_{i-1},1)$ to compute $\widetilde{q}_i$ via (8). To obtain $\widehat{u}_{i+1}(x_1,\dots,x_{i-1},0)$, we find a mincut $\gamma_i^-=\gamma_i$ as above for $x_1,\dots,x_{i-1}$ fixed and $x_i=0$, and put $\widehat{u}_{i+1}(x_1,\dots,x_{i-1},0)=\mathbb{P}[E(\gamma_i^-)]$. We do the same with $x_i=1$ to find a mincut $\gamma_i^+=\gamma_i$, and put $\widehat{u}_{i+1}(x_1,\dots,x_{i-1},1)=\mathbb{P}[E(\gamma_i^+)]$. Note that these mincuts can be computed in polynomial time [29] (a typical way of doing this calculation is by defining as the cost of link i the number $-\ln q_i$). Then we generate X_i using this probability $\widetilde{q}_i$, and we compute the corresponding term of the likelihood ratio.

Of course, these computations add overhead to the algorithm compared with crude MC, but this computational burden can be more than compensated by the huge variance reduction, as our experiments will show.

Our overall procedure based on this mincut-maxprob approximation is given in Algorithm 2 below. This algorithm computes one realization of the estimator $Y = \psi(X)L(X)$, given the graph $\mathcal{G}$, and its link unreliabilities q_i. In practice, one would compute n s-independent realizations of Y, estimate the unreliability u by their average, and compute a confidence interval on u in the usual way.

Algorithm 2 Approximate Zero-Variance Simulation with Minimal Cuts

$L \leftarrow 1$;
for $i = 1$ to m **do**
  find a mincut γ_i^- of maximal probability in the set $\mathcal{C}_i$ that corresponds to $\mathcal{G}_i(x_1, \ldots, x_{i-1}, 0)$;
  $\widehat{u}_{i+1}(x_1, \ldots, x_{i-1}, 0) \leftarrow \mathbb{P}[E(\gamma_i^-)]$;
  find a mincut γ_i^+ of maximal probability in the set $\mathcal{C}_i$ that corresponds to $\mathcal{G}_i(x_1, \ldots, x_{i-1}, 1)$;
  $\widehat{u}_{i+1}(x_1, \ldots, x_{i-1}, 1) \leftarrow \mathbb{P}[E(\gamma_i^+)]$;
  compute $\widetilde{q}_i$ via (8);
  generate U_i a uniform random variate over $(0, 1)$;
  if $U_i < \widetilde{q}_i$ **then**
    $x_i \leftarrow 0$; $L_i \leftarrow q_i/\widetilde{q}_i$;
  else
    $x_i \leftarrow 1$; $L_i \leftarrow (1 - q_i)/(1 - \widetilde{q}_i)$;
  end if
  $L \leftarrow L \times L_i$;
end for
return $Y = \psi(x_1, \ldots, x_m) \times L$;

B. Asymptotic Analysis

We now show that the proposed mincut-maxprob approximation always gives an estimator with BRE.

In the parameterized setting where the q_i depend on ϵ, the unreliabilities of the mincuts obviously depend on ϵ, and the set of mincuts with maximal probability may change as a function of ϵ. We show below that, when $\epsilon \to 0$, this set eventually becomes fixed. This limiting set will be called the set of mincuts with *asymptotically maximal probability*. Then, for the asymptotic analysis, we will assume that ϵ is small enough so that the mincuts with maximal probability are the same as the mincuts with asymptotically maximal probability.

Under the assumption that the edge unreliabilities have the form (1), for any $\gamma \in \mathcal{C}$, we have

$$\mathbb{P}[E(\gamma)] = \prod_{i \in \gamma} q_i = a_\gamma \epsilon^{c_\gamma}, \tag{16}$$

where

$$a_\gamma = \prod_{i \in \gamma} a_i \quad \text{and} \quad c_\gamma = \sum_{i \in \gamma} b_i. \tag{17}$$

Lemma 1: For a model where the graph is connected, and (1) holds, we have $u = \Theta(\epsilon^c)$ with $c = \min_{\gamma \in \mathcal{C}} c_\gamma$. We also have $c > 0$ (the system's failure is a rare event) iff every mincut contains at least one link i with $b_i > 0$.

Proof: The inclusion-exclusion formula applied to u immediately leads to the first result. We also have $c > 0$ iff $c_\gamma > 0$ for all $\gamma \in \mathcal{C}$, iff every mincut γ contains a link i with $b_i > 0$. ■

The set of mincuts having maximal probability obviously depends on ϵ, as illustrated by the next example.

Example 1: Consider the graph represented in Fig. 1, where we want to compute the probability that the gray nodes A and D are disconnected. Suppose the unreliabilities of the links are $q_1 = q_2 = \epsilon$, and $q_3 = q_4 = 20\epsilon^2$. This model has four mincuts that we can denote as $\gamma_{1,2}$, $\gamma_{1,4}$, $\gamma_{2,3}$, and $\gamma_{3,4}$, using as indexes the links composing the cut. We have $\mathbb{P}[E(\gamma_{1,2})] = \epsilon^2$, $\mathbb{P}[E(\gamma_{1,4})] = \mathbb{P}[E(\gamma_{2,3})] = 20\epsilon^3$, $\mathbb{P}[E(\gamma_{3,4})] = 400\epsilon^4$. When $\epsilon = 0.1$, we have $\mathbb{P}[E(\gamma_{1,2})] = 0.01 < \mathbb{P}[E(\gamma_{1,4})] = \mathbb{P}[E(\gamma_{2,3})] = 0.02 < \mathbb{P}[E(\gamma_{3,4})] = 0.04$, while if $\epsilon < 1/20$, the order is $\mathbb{P}[E(\gamma_{1,2})] > \mathbb{P}[E(\gamma_{1,4})] = \mathbb{P}[E(\gamma_{2,3})] > \mathbb{P}[E(\gamma_{3,4})]$.

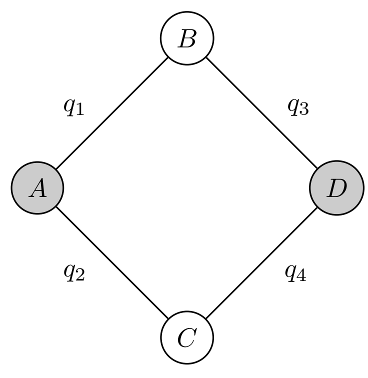

Fig. 1. A graph topology with four links, and two nodes (A and D) that require to be connected.

Consider the set of cuts γ for which $c_\gamma = c$ (the smallest exponent), and among them, take those having the largest a_γ, say $a_\gamma = a_0$. Let

$$\epsilon_0 = \max\left\{\epsilon \in \mathbb{R} \text{ such that } a_0\epsilon^c \geq a_\gamma \epsilon^{c_\gamma} \text{ for all } \gamma \in \mathcal{C}\right\}.$$

Lemma 2: For a connected model where (1) holds, we have $\epsilon_0 > 0$; and for all $\epsilon \leq \epsilon_0$, the set of mincuts with maximal probability is exactly the set of mincuts with probability $a_0\epsilon^c$.

Proof: The fact that $\epsilon_0 > 0$ follows from the fact that there are a finite number of mincuts, and our definitions of c and a_0. The rest is immediate from the definition of ϵ_0. ■

The previous lemmas also apply to any subgraph $\mathcal{G}_i$ obtained at step i of the algorithm. In particular, we have

$$u_{i+1}(x_1,\ldots,x_i) = \Theta(\epsilon^{c_{i+1}(x_1,\ldots,x_i)}) \qquad (18)$$

where

$$c_{i+1}(x_1,\ldots,x_i) = \min\{d: \mathbb{P}[E(\gamma)] = \Theta(\epsilon^d) \text{ for some mincut } \gamma \text{ of } \mathcal{G}_i(x_1,\ldots,x_i)\ \}.$$

The constants $c_{i+1}(x_1,\ldots,x_i) \geq 0$ are independent of ϵ, but depend on the subgraph $\mathcal{G}_i$.

We are now ready to prove the main theorems on our proposed approximation.

Theorem 4: The mincut-maxprob approximation described above always satisfies the condition of Theorem 2. Consequently, it gives an IS scheme with the BRE property.

Proof: Fixing the state of links 1 to i to $x_1,\ldots,x_i$, we just saw that there exists a constant $c_{i+1}(x_1,\ldots,x_i) \geq 0$ independent of ϵ such that $u_{i+1}(x_1,\ldots,x_i) = \Theta(\epsilon^{c_{i+1}(x_1,\ldots,x_i)})$. As a consequence, the mincut-maxprob approximation gives

$$\begin{aligned}\widehat{u}_{i+1}(x_1,\ldots,x_i) &= \Theta(\epsilon^{c_{i+1}(x_1,\ldots,x_i)})\\ &= \Theta(u_{i+1}(x_1,\ldots,x_i)).\end{aligned}$$

The assumptions of Theorem 2 are thus verified. ■

Theorem 5: Under an IS scheme based on the mincut-maxprob approximation, suppose that for each $x = (x_1,\ldots,x_m) \in \mathcal{S}_1$, and for each $i \in \{1,\ldots,m-1\}$, the graph $\mathcal{G}_i(x_1,\ldots,x_i)$ contains only one mincut with probability $\Theta(u_{i+1}(x_1,\ldots,x_i))$, and one of the following three conditions is satisfied:

(i) the graph $\mathcal{G}_i(x_1,\ldots,x_{i-1},1-x_i)$ contains only one mincut having probability $\Theta(u_{i+1}(x_1,\ldots,x_{i-1},1-x_i))$, or

(ii) $x_i = 0$ and

$$(1-q_i)\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},1) = o(q_i\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},0)), \qquad (19)$$

or

(iii) $x_i = 1$ and

$$q_i\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},0) = o((1-q_i)\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},1)). \qquad (20)$$

Then the assumptions of Theorem 3 are satisfied, and consequently we have an IS scheme with the VRE property.

Proof: Because there is only one mincut with probability $\Theta(u_i(x_1,\ldots,x_{i-1}))$ in $\mathcal{G}_{i-1}$, the (finite) sum of probabilities of the other cuts is $o(u_i(x_1,\ldots,x_{i-1}))$. Thus for all $i \in \{1,\ldots,m-1\}$,

$$\frac{\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},x_i)}{u_{i+1}(x_1,\ldots,x_{i-1},x_i)} = 1 + o(1).$$

Similarly, the condition that there is also one mincut with probability $\Theta(u_{i+1}(x_1,\ldots,1-x_i))$ corresponds to

$$\frac{\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},1-x_i)}{u_{i+1}(x_1,\ldots,x_{i-1},1-x_i)} = 1 + o(1),$$

implying that (10) holds, while conditions (19) and (20) correspond exactly to (11) and (12). Then the result follows from Theorem 3. ■

Our next and final result gives sufficient VRE conditions for the special case where $\mathcal{G}$ has homogeneous link reliabilities.

Theorem 6: Suppose that the graph $\mathcal{G}$ is homogeneous, which means (without loss of generality) that $q_i = \epsilon$ for all i. Suppose also that we have an IS scheme based on the mincut-maxprob approximation, under which for each $x \in \mathcal{S}_1$ and each $i \in \{1,\ldots,m-1\}$, one of the following two conditions holds:

(i) the number of mincuts of probability $\Theta(u_{i+1}(x_1,\ldots,x_{i-1},0))$ in $\mathcal{G}_i(x_1,\ldots,x_{i-1},0)$ is the same as the number of mincuts of probability $\Theta(u_{i+1}(x_1,\ldots,x_{i-1},1))$ in $\mathcal{G}_i(x_1,\ldots,x_{i-1},1)$; or

(ii) the graph $\mathcal{G}_i(x_1,\ldots,x_i)$ contains only one mincut with probability $\Theta(u_{i+1}(x_1,\ldots,x_i))$, and either $x_i = 0$ and

$$(1-q_i)\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},1) = o(q_i\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},0)), \qquad (21)$$

or $x_i = 1$ and

$$q_i\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},0) = o((1-q_i)\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},1)). \qquad (22)$$

Then the assumptions of Theorem 3 are satisfied, and consequently we have an IS scheme with the VRE property.

Proof: Let $f_{i+1}(x_1,\ldots,x_i)$ be the number of mincuts with probability $\Theta(u_{i+1}(x_1,\ldots,x_i))$. Under the first condition, we have $f_{i+1}(x_1,\ldots,x_{i-1},0) = f_{i+1}(x_1,\ldots,x_{i-1},1)$. Similar to the proof of Theorem 4, $u_{i+1}(x_1,\ldots,x_i) = f_{i+1}(x_1,\ldots,x_i)\epsilon^{c_{i+1}(x_1,\ldots,x_i)} + o(u_{i+1}(x_1,\ldots,x_i))$ while $\widehat{u}_{i+1}(x_1,\ldots,x_i) = \epsilon^{c_{i+1}(x_1,\ldots,x_i)}$. Then,

$$\begin{aligned}\frac{\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},1)}{u_{i+1}(x_1,\ldots,x_{i-1},1)} &= \frac{1}{f_{i+1}(x_1,\ldots,x_{i-1},0)} + o(1),\\ \frac{\widehat{u}_{i+1}(x_1,\ldots,x_{i-1},0)}{u_{i+1}(x_1,\ldots,x_{i-1},0)} &= \frac{1}{f_{i+1}(x_1,\ldots,x_{i-1},1)} + o(1),\end{aligned}$$

which corresponds to (10). In the second set of conditions, (21), and (22) correspond to (11), and (12), respectively. The conditions of Theorem 3 are therefore verified. ■

C. Examples

We now give small examples to illustrate the mincut-maxprob approximation. In all our examples, the links are sampled by order of their number. In Example 2 the mincut-maxprob approximations correspond to the exact unreliabilities, and the estimator has zero variance. Example 3 illustrates a situation where the mincut-maxprob approximations are not always exact, but the overall estimator still has zero-variance. In Example 4, we have VRE; while in Example 5, we have BRE but not VRE. Note that, for the three examples where we have VRE, one can verify by going through the details that VRE also holds for any ordering of the links. In these examples, the mincuts used in our approximation are always those with asymptotically maximal probability.

Example 2: Consider the tiny graph of Fig. 2. The aim is to compute the probability that the gray nodes A and C are disconnected. The links are homogeneous with unreliability $q_i = \epsilon$ for $i = 1, 2, 3$, for some small real number ϵ. For this model, only three configurations lead

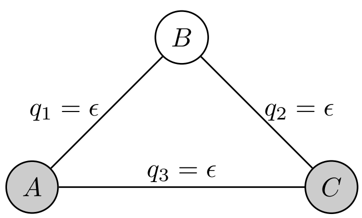

Fig. 2. A graph with three links.

to disconnected nodes: $(0, 0, 0)$, $(0, 1, 0)$, and $(1, 0, 0)$. The system unreliability is $u = \epsilon^3 + 2\epsilon^2(1-\epsilon) = 2\epsilon^2 - \epsilon^3$. We can check that $u_2(0) = \epsilon$, $u_2(1) = \epsilon^2$, $u_3(0, 0) = \epsilon$, $u_3(0, 1) = \epsilon$, $u_3(1, 0) = \epsilon$, and $u_3(1, 1) = 0$. At step 1, we need to compute $\widehat{u}_2(0)$ and $\widehat{u}_2(1)$ to find $\widetilde{q}_1$, and sample the first link. For $x_1 = 0$, the mincut γ_1^- contains only link 3, and its failure probability is $\widehat{u}_2(0) = \epsilon = u_2(0)$. For $x_1 = 1$, the mincut γ_1^+ contains links 2 and 3, and we have $\widehat{u}_2(1) = \epsilon^2 = u_2(1)$. This result gives

$$\widetilde{q}_1 = \frac{q_1 \widehat{u}_2(0)}{q_1 \widehat{u}_2(0) + (1-q_1)\widehat{u}_2(1)} = \frac{\epsilon^2}{\epsilon^2 + (1-\epsilon)\epsilon^2} = \frac{1}{2-\epsilon}.$$

Continuing in this way, we get $\widehat{u}_3(0, 0) = \epsilon$, $\widehat{u}_3(0, 1) = \epsilon$, $\widehat{u}_3(1, 0) = \epsilon$, and $\widehat{u}_3(1, 1) = 0$, which are all equal to the corresponding values of $u_i(\cdot)$. We have $\widetilde{q}_2 = \epsilon$ when $x_1 = 0$, and $\widetilde{q}_2 = 1$ when $x_1 = 1$. The values of u_4, and $\widehat{u}_4$ are also both equal to those of ψ; and we have $\widetilde{q}_3 = 1$ on all paths having nonzero probability. Fig. 3 gives a transition diagram of the Markov chain under this IS sampling scheme. The first node on the left represents the initial state, and the nodes on the far right (the leaves) are the final states (or configurations). The states in gray have zero probability of being reached. In this example, we have $\hat{u}_i = u_i$ everywhere, and consequently our approximate zero-variance estimator is exactly the same as the zero-variance one.

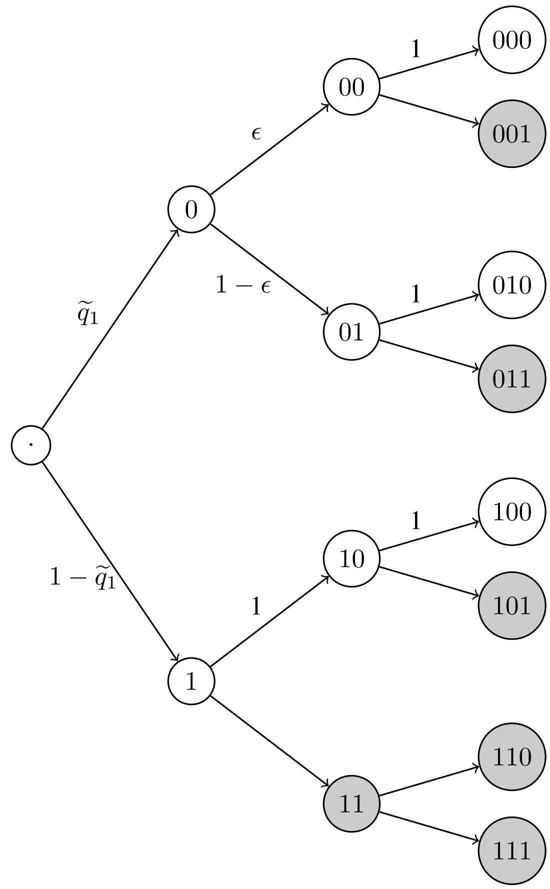

Fig. 3. Possible transitions of the Markov chain for Example 2.

Example 3: Consider again the graph of Example 1, Fig. 1, where we want to compute the probability that the gray nodes A and D are disconnected. The links are assumed homogeneous, with unreliabilities $q_i = \epsilon$ for $i = 1, \ldots, 4$. There are 16 possible configurations of the 4 links, and one can verify that A and D are disconnected for 9 of them, and that the unreliability is $u = 4\epsilon^2 + o(\epsilon^2)$. The reader can verify in this case that $u_2(0) = 2\epsilon - \epsilon^2$, $u_2(1) = 2\epsilon^2 - \epsilon^3$, $u_3(0, 0) = 1$, $u_3(0, 1) = \epsilon$, $u_3(1, 0) = \epsilon$, and $u_3(1, 1) = \epsilon^2$, for instance, whereas the mincut-maxprob approximation gives $\widehat{u}_2(0) = \epsilon$, $\widehat{u}_2(1) = \epsilon^2$, $\widehat{u}_3(0, 0) = 1$, $\widehat{u}_3(0, 1) = \epsilon$, $\widehat{u}_3(1, 0) = \epsilon$, and $\widehat{u}_3(1, 1) = \epsilon^2$. The values of $\widehat{u}_4$ are exactly the same as those of u_4. Observe that $\widehat{u}_2(0) \approx u_2(0)/2$ and $\widehat{u}_2(1) \approx u_2(1)/2$, whereas $\widehat{u}_i(\cdot) = u_i(\cdot)$ in the other cases. Thus, the conditions of Theorem 2 are satisfied, but not with $a_{i+1}(x_1,\dots,x_i) = 1$ for all i, because $a_2(0) = a_2(1) = 1/2$. However, the conditions of Theorem 3 are also satisfied, because $\widehat{u}_2(0)/u_2(0) = \widehat{u}_2(1)/u_2(1)+o(1)$, so we have VRE. In fact, one can also verify for this particular example that the IS scheme with our approximation gives exactly zero variance, even though $\hat{u}_2 \neq u_2$. The reason is that even when sampling the first link (using $\hat{u}_2$), the mincut-maxprob approximation yields

$$\begin{aligned}\widetilde{q}_1 &= \frac{q_1\widehat{u}_2(0)}{q_1\widehat{u}_2(0)+(1-q_1)\widehat{u}_2(1)} \\ &= \frac{\epsilon^2}{\epsilon^2+(1-\epsilon)\epsilon^2} = \frac{1}{2-\epsilon},\end{aligned}$$

which is exactly the same as when using zero-variance IS:

$$\begin{aligned}\widetilde{q}_1 &= \frac{q_1 u_2(0)}{q_1 u_2(0)+(1-q_1)u_2(1)} \\ &= \frac{\epsilon(2\epsilon-\epsilon^2)}{\epsilon(2\epsilon-\epsilon^2)+(1-\epsilon)(2\epsilon^2-\epsilon^3)} = \frac{1}{2-\epsilon}.\end{aligned}$$

The next two examples illustrate a situation where Theorems 5 applies and we have VRE, and then another situation where only BRE holds.

Example 4: Consider the graph of Fig. 4, where we again want to compute the probability that the gray nodes A and D are disconnected. Links are assumed homogeneous, with unreliability $q_i = \epsilon$ for $i = 1,\dots,5$.

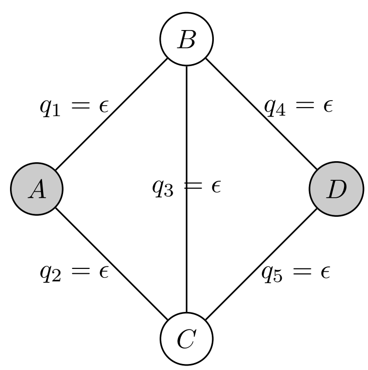

Fig. 4. Graph topology with five links and two nodes requiring to be connected.

Table I compares the unreliabilities and mincut-maxprob approximations at the different steps of the method. To simplify, we only list the states in which the final outcome (being connected or not) is not yet known. Table II, on the other hand, lists the 16 out of 32 configurations leading to disconnected nodes A and D, with their probabilities under the original probability law, and under IS. Only two configurations, $(0,0,1,1,1)$ and $(1,1,1,0,0)$, have a probability $\Theta(1)$ under IS, and it can be readily checked that the sufficient conditions of Theorem 6 are verified. To verify

TABLE I
UNRELIABILITIES AND CORRESPONDING MINCUT-MAXPROB APPROXIMATIONS FOR EXAMPLE 4.

State $(x_1,\dots x_i)$	$u_{i+1}(x_1,\dots x_i)$	$\widehat{u}_{i+1}(x_1,\dots x_i)$
(0)	$\epsilon+o(\epsilon)$	ϵ
(1)	$\epsilon^2+o(\epsilon^2)$	ϵ^2
$(0,0)$	1	1
$(0,1)$	$2\epsilon^2-\epsilon^3$	ϵ^2
$(1,0)$	$2\epsilon^2-\epsilon^3$	ϵ^2
$(1,1)$	ϵ^2	ϵ^2
$(0,1,0)$	ϵ	ϵ
$(0,1,1)$	ϵ^2	ϵ^2
$(1,0,0)$	ϵ	ϵ
$(1,0,1)$	ϵ^2	ϵ^2
$(1,1,0)$	ϵ^2	ϵ^2
$(1,1,1)$	ϵ^2	ϵ^2
$(0,1,0,0)$	ϵ	ϵ
$(0,1,0,1)$	ϵ	ϵ
$(0,1,1,0)$	ϵ	ϵ
$(0,1,1,1)$	0	0
$(1,0,0,0)$	1	1
$(1,0,0,1)$	0	0
$(1,0,1,0)$	ϵ	ϵ
$(1,0,1,1)$	0	0
$(1,1,0,0)$	ϵ	ϵ
$(1,1,0,1)$	0	0
$(1,1,1,0)$	ϵ	ϵ
$(1,1,1,1)$	0	0

VRE directly, Table II also displays the contribution of each configuration to the variance of the overall estimator under IS. We find from this table and easy computations that $u = 2\epsilon^2 + 2\epsilon^3 - 5\epsilon^4 + 2\epsilon^5$, and

$$\sum_{x\in\{0,1\}^5} L^2(x)\psi(x)\widetilde{\mathbb{P}}[X=x] = 4\epsilon^4 + o(\epsilon^4),$$

so the relative variance is

$$\frac{1}{u^2}\sum_{x\in\{0,1\}^5} \psi(x)L^2(x)\widetilde{\mathbb{P}}[X=x] - 1 = o(1),$$

in accordance with Theorem 6.

Example 5: Consider a variation of Example 3 where $\mathcal{K}$ is now the set of all nodes, as shown in Fig. 5. The system works only if all nodes are connected. It is in a failed state iff at least two links are failed. This fact gives 11 failed configurations out of 16, and $u = 6\epsilon^2 + o(\epsilon^2)$. Table III gives the original, and modified probabilities of those 11 configurations, as well as the contributions to the second moment under IS. Here, for instance, $u_3(0,0) = \widehat{u}_3(0,0) = 1$, but $u_3(0,1) = 2\epsilon + o(\epsilon)$ whereas $\widehat{u}_3(0,1) = \epsilon$. Thus, if $x_1 = 0$, the unreliability of the second link under the zero-variance change of measure is $\epsilon u_3(0,0)/(\epsilon u_3(0,0)+(1-\epsilon)u_3(0,1)) = 1/3 + o(1)$, and it is $1/2 + o(1)$ with the mincut-maxprob approximation. Thus we have BRE, thanks to Theorem 4, but the sufficient conditions of Theorems 5 or 6 are not verified, and this can be seen by looking for instance at configuration $(0,0,1,1)$.

TABLE II
PROBABILITIES OF THE 16 CONFIGURATIONS X THAT GIVE $\psi(X) = 1$ UNDER THE ORIGINAL AND MODIFIED PROBABILITIES, AND THE CORRESPONDING CONTRIBUTIONS TO THE SECOND MOMENT UNDER IS, FOR EXAMPLE 4.

Configuration	$\mathbb{P}[X = x]$	$\widetilde{\mathbb{P}}[X = x]$	$L^2(x)\widetilde{\mathbb{P}}[X = x]$
$(0,0,0,0,0)$	ϵ^5	$\frac{\epsilon^3}{(2-\epsilon)(1+\epsilon-\epsilon^2)}$	$\epsilon^7(2-\epsilon)(1+\epsilon-\epsilon^2)$
$(0,0,0,0,1)$	$\epsilon^4(1-\epsilon)$	$\frac{\epsilon^2(1-\epsilon)}{(2-\epsilon)(1+\epsilon-\epsilon^2)}$	$\epsilon^6(1-\epsilon)(2-\epsilon)(1+\epsilon-\epsilon^2)$
$(0,0,0,1,0)$	$\epsilon^4(1-\epsilon)$	$\frac{\epsilon^2(1-\epsilon)}{(2-\epsilon)(1+\epsilon-\epsilon^2)}$	$\epsilon^6(1-\epsilon)(2-\epsilon)(1+\epsilon-\epsilon^2)$
$(0,0,0,1,1)$	$\epsilon^3(1-\epsilon)^2$	$\frac{\epsilon(1-\epsilon)^2}{(2-\epsilon)(1+\epsilon-\epsilon^2)}$	$\epsilon^5(1-\epsilon)^2(2-\epsilon)(1+\epsilon-\epsilon^2)$
$(0,0,1,0,0)$	$\epsilon^4(1-\epsilon)$	$\frac{\epsilon^2(1-\epsilon)}{(2-\epsilon)(1+\epsilon-\epsilon^2)}$	$\epsilon^6(1-\epsilon)(2-\epsilon)(1+\epsilon-\epsilon^2)$
$(0,0,1,1,0)$	$\epsilon^3(1-\epsilon)^2$	$\frac{\epsilon(1-\epsilon)^2}{(2-\epsilon)(1+\epsilon-\epsilon^2)}$	$\epsilon^5(1-\epsilon)^2(2-\epsilon)(1+\epsilon-\epsilon^2)$
$(0,0,1,0,1)$	$\epsilon^3(1-\epsilon)^2$	$\frac{\epsilon(1-\epsilon)^2}{(2-\epsilon)(1+\epsilon-\epsilon^2)}$	$\epsilon^5(1-\epsilon)^2(2-\epsilon)(1+\epsilon-\epsilon^2)$
$(0,0,1,1,1)$	$\epsilon^2(1-\epsilon)^3$	$\frac{(1-\epsilon)^3}{(2-\epsilon)(1+\epsilon-\epsilon^2)}$	$\epsilon^4(1-\epsilon)^3(2-\epsilon)(1+\epsilon-\epsilon^2)$
$(0,1,0,0,0)$	$\epsilon^4(1-\epsilon)$	$\frac{\epsilon^2(1-\epsilon)}{(2-\epsilon)^2(1+\epsilon-\epsilon^2)}$	$\epsilon^6(1-\epsilon)(2-\epsilon)^2(1+\epsilon-\epsilon^2)$
$(0,1,0,1,0)$	$\epsilon^3(1-\epsilon)^2$	$\frac{\epsilon(1-\epsilon)^2}{(2-\epsilon)^2(1+\epsilon-\epsilon^2)}$	$\epsilon^5(1-\epsilon)^2(2-\epsilon)^2(1+\epsilon-\epsilon^2)$
$(0,1,1,0,0)$	$\epsilon^3(1-\epsilon)^2$	$\frac{\epsilon(1-\epsilon)^2}{(2-\epsilon)^2(1+\epsilon-\epsilon^2)}$	$\epsilon^5(1-\epsilon)^2(2-\epsilon)^2(1+\epsilon-\epsilon^2)$
$(1,0,0,0,0)$	$\epsilon^4(1-\epsilon)$	$\frac{\epsilon^2(1-\epsilon)}{(2-\epsilon)^2}$	$\epsilon^6(1-\epsilon)(2-\epsilon)^2$
$(1,0,0,0,1)$	$\epsilon^3(1-\epsilon)^2$	$\frac{\epsilon(1-\epsilon)^2}{(2-\epsilon)^2}$	$\epsilon^5(1-\epsilon)^2(2-\epsilon)^2$
$(1,0,1,0,0)$	$\epsilon^3(1-\epsilon)^2$	$\frac{\epsilon(1-\epsilon)^2}{(2-\epsilon)^2}$	$\epsilon^5(1-\epsilon)^2(2-\epsilon)^2$
$(1,1,0,0,0)$	$\epsilon^3(1-\epsilon)^2$	$\frac{\epsilon(1-\epsilon)^2}{(2-\epsilon)}$	$\epsilon^5(1-\epsilon)^2(2-\epsilon)$
$(1,1,1,0,0)$	$\epsilon^2(1-\epsilon)^3$	$\frac{(1-\epsilon)^3}{(2-\epsilon)}$	$\epsilon^4(1-\epsilon)^2(2-\epsilon)$

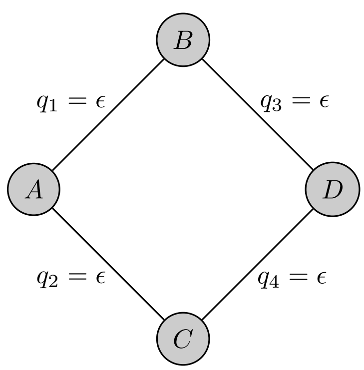

Fig. 5. A graph topology with four links, and requiring full connectivity.

One can directly see that VRE is not verified by directly computing the relative variance:

$$\frac{1}{u^2} \sum_{x \in \{0,1\}^m} \psi(x) L^2(x) \widetilde{\mathbb{P}}[X = x] - 1 = \frac{1}{9} + o(1).$$

D. How to order the links

We mentioned earlier that the visiting order of the links may have an impact on the performance of our proposed algorithm, and that we do not have a particular ordering method to propose at this time. We emphasize that the BRE property is always verified, regardless of the order in which the links are sampled. For the VRE property, on the other hand, the sampling order may matter. Thus, the ordering may have an impact on the performance of the estimation procedure, both in terms of variance and execution time. However, in all the examples that we have tried, it turned out that the ordering never mattered much.

A similar ordering issue occurs in combinatorial algorithms that perform an exact computation of network unreliability. It happens, for instance, with the method in [1], or with the best combinatorial approach, called "factoring", proposed by Satyayarana and Chang in [28]; see the comments given in [25]. The execution time of these types of combinatorial algorithms depends (sometimes strongly) on the visiting order on the model's components, but so far there is no available result for understanding this dependency, and to support a particular ordering method.

VI. NUMERICAL ILLUSTRATIONS

We now report some simulation experiments with the mincut-maxprob approximation, for the toy examples of Section V-C, and for larger ones. We also tried the method numerically for Examples 2, and 3; and the estimator was equal to u on each run, as expected. We compare the results with those obtained by the path-based (PB) method of [8] for the same examples. That PB method is the only other technique for which BRE has been proved,

TABLE III
ORIGINAL, AND MODIFIED PROBABILITIES OF THE 11 FAILED CONFIGURATIONS, AND ASSOCIATED CONTRIBUTION TO THE SECOND MOMENT UNDER IS, FOR EXAMPLE 5.

Configuration	$\mathbb{P}[X = x]$	$\widetilde{\mathbb{P}}[X = x]$	$L^2(x)\widetilde{\mathbb{P}}[X = x]$
$(0, 0, 0, 0)$	ϵ^4	$\frac{\epsilon^2}{(2-\epsilon)^2}$	$\epsilon^6(2-\epsilon)^2$
$(0, 0, 0, 1)$	$\epsilon^3(1-\epsilon)$	$\frac{\epsilon(1-\epsilon)}{(2-\epsilon)^2}$	$\epsilon^5(1-\epsilon)(2-\epsilon)^2$
$(0, 0, 1, 0)$	$\epsilon^3(1-\epsilon)$	$\frac{\epsilon(1-\epsilon)}{(2-\epsilon)^2}$	$\epsilon^5(1-\epsilon)(2-\epsilon)^2$
$(0, 0, 1, 1)$	$\epsilon^2(1-\epsilon)^2$	$\frac{(1-\epsilon)^2}{(2-\epsilon)^2}$	$\epsilon^4(1-\epsilon)^2(2-\epsilon)^2$
$(0, 1, 0, 0)$	$\epsilon^3(1-\epsilon)$	$\frac{\epsilon(1-\epsilon)}{(2-\epsilon)^3}$	$\epsilon^5(1-\epsilon)(2-\epsilon)^3$
$(0, 1, 0, 1)$	$\epsilon^2(1-\epsilon)^2$	$\frac{(1-\epsilon)^2}{(2-\epsilon)^3}$	$\epsilon^4(1-\epsilon)^2(2-\epsilon)^3$
$(0, 1, 1, 0)$	$\epsilon^2(1-\epsilon)^2$	$\frac{(1-\epsilon)^2}{(2-\epsilon)^3}$	$\epsilon^4(1-\epsilon)^2(2-\epsilon)^3$
$(1, 0, 0, 0)$	$\epsilon^3(1-\epsilon)$	$\frac{\epsilon(1-\epsilon)}{(2-\epsilon)^3}$	$\epsilon^5(1-\epsilon)(2-\epsilon)^3$
$(1, 0, 0, 1)$	$\epsilon^2(1-\epsilon)^2$	$\frac{(1-\epsilon)^2}{(2-\epsilon)^3}$	$\epsilon^4(1-\epsilon)^2(2-\epsilon)^3$
$(1, 0, 1, 0)$	$\epsilon^2(1-\epsilon)^2$	$\frac{(1-\epsilon)^2}{(2-\epsilon)^3}$	$\epsilon^4(1-\epsilon)^2(2-\epsilon)^3$
$(1, 1, 0, 0)$	$\epsilon^2(1-\epsilon)^2$	$\frac{(1-\epsilon)^2}{(2-\epsilon)^2}$	$\epsilon^4(1-\epsilon)^2(2-\epsilon)^2$

and only under a link homogeneity assumption (all links must have the same unreliability), which is not required for our new proposal (it always has the BRE property). It was also shown via a counterexample in [8] that BRE does not always hold if the links are not homogeneous. We will make the comparison under this link homogeneity assumption, where PB is guaranteed to have the BRE property.

Example 6: Table IV gives empirical results for Example 4, for sample size $n = 10^4$, and four different values of ϵ. It shows the estimate of u, a 95% confidence interval on u, the empirical standard deviation per replication (STD), and the empirical relative error (RE). We observe a decrease of the relative error when ϵ decreases, in accordance with the VRE property. The PB method on the other hand does not satisfy VRE, but only BRE, and this shows in the empirical results: the variance of the PB estimator is always significantly larger.

Example 7: Table V reports similar results for Example 5. We saw in that example that the square relative error is $1/9 + o(1)$, so the relative error should converge to 1/3 when $\epsilon \to 0$. This is exactly what we observe empirically. We do not compare here with the PB method, because we do not have an implementation dealing with the full connectivity estimation.

Example 8: We now take a larger graph, made of 20 nodes and 30 links, with the dodecahedron topology as shown in Fig. 6. This structure is often used as a benchmark for network reliability evaluation techniques [7], [8], [14]. We consider the homogeneous case, where all links have the same unreliability ϵ, and we want to compute the probability that nodes A and B are disconnected. Links are ordered somewhat arbitrarily, according to their numbering in the figure. The empirical results, in Table VI, suggest that VRE holds. Our proposed IS scheme provides an extremely tight confidence interval on u when ϵ is very small. We know of no other method that provides VRE for this dodecahedron topology. Here too, our proposal yields significantly better results than the PB method.

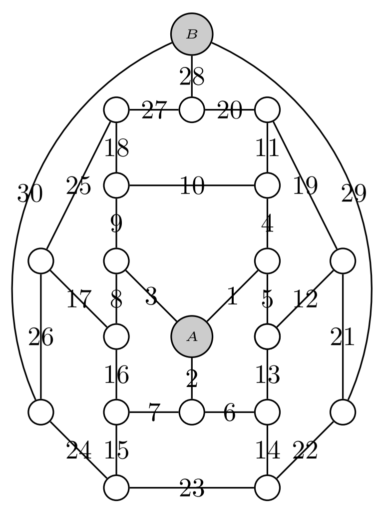

Fig. 6. Dodecahedron topology.

Example 9: To experiment with an even larger network, of a size for which simulation is typically required, we construct a graph by juxtaposing three copies of the dodecaheron topology of Fig. 6, forming a parallel system as shown in Fig. 7. The resulting network has 90 links. The source and terminal nodes A and B of the three dodecaheron copies are merged, and the links are ordered in the same way as in Fig. 6, starting with the first copy,

TABLE IV
EMPIRICAL RESULTS FOR EXAMPLE 6, FOR $n = 10^4$, AND FOUR VALUES OF ϵ. RESPECTIVELY STD, AND RE REPRESENT THE EMPIRICAL STANDARD DEVIATION, AND RELATIVE ERROR FOR OUR PROPOSED METHOD, WHILE SDT-PB, AND RE-PB ARE THOSE FOR THE PB METHOD.

ϵ	Estimate	95% confidence interval	STD	RE	STD-PB	RE-PB
10^{-1}	2.150×10^{-2}	$(2.139 \times 10^{-2},\ 2.160 \times 10^{-2})$	5.15×10^{-3}	0.24	1.773×10^{-2}	0.82
10^{-2}	2.022×10^{-4}	$(2.018 \times 10^{-4},\ 2.026 \times 10^{-4})$	2.07×10^{-5}	0.10	1.980×10^{-4}	0.98
10^{-3}	2.0014×10^{-6}	$(2.0003 \times 10^{-6},\ 2.0024 \times 10^{-6})$	5.28×10^{-8}	0.026	1.998×10^{-6}	1.0
10^{-4}	2.0002×10^{-8}	$(1.9998 \times 10^{-8},\ 2.0006 \times 10^{-8})$	2.00×10^{-10}	0.010	2.000×10^{-8}	1.0

TABLE V
EMPIRICAL RESULTS FOR EXAMPLE 7, FOR $n = 10^4$ AND FOUR VALUES OF ϵ.

ϵ	Estimate	95% confidence interval	STD	RE
10^{-1}	5.271×10^{-2}	$(5.239 \times 10^{-2},\ 5.302 \times 10^{-2})$	1.62×10^{-2}	0.31
10^{-2}	5.944×10^{-4}	$(5.905 \times 10^{-4},\ 5.982 \times 10^{-4})$	1.96×10^{-4}	0.33
10^{-3}	6.015×10^{-6}	$(5.976 \times 10^{-6},\ 6.054 \times 10^{-6})$	2.00×10^{-6}	0.33
10^{-4}	6.022×10^{-8}	$(5.982 \times 10^{-8},\ 6.061 \times 10^{-8})$	2.00×10^{-8}	0.33

TABLE VI
EMPIRICAL RESULTS FOR EXAMPLE 8, FOR $n = 10^4$, AND FOUR VALUES OF ϵ.

ϵ	Estimate	95% confidence interval	STD	RE	STD-PB	RE-PB
10^{-1}	2.8960×10^{-3}	$(2.8276 \times 10^{-3},\ 2.9645 \times 10^{-3})$	3.49×10^{-3}	1.2	1.59×10^{-2}	5.5
10^{-2}	2.0678×10^{-6}	$(2.0611 \times 10^{-6},\ 2.0744 \times 10^{-6})$	3.42×10^{-7}	0.17	1.924×10^{-5}	9.3
10^{-3}	2.0076×10^{-9}	$(2.0053 \times 10^{-9},\ 2.0099 \times 10^{-9})$	1.14×10^{-10}	0.057	1.980×10^{-8}	9.9
10^{-4}	2.0007×10^{-12}	$(2.0000 \times 10^{-12},\ 2.0014 \times 10^{-12})$	3.46×10^{-14}	0.017	1.993×10^{-11}	9.9

then the second copy, and finally the third. We still consider the homogeneous case, where all links have the same unreliability ϵ, and we want to compute the probability that nodes A and B are disconnected. Note that the unreliability here is the cube of that of a single dodecahedron. The empirical results, in Table VII, suggest that VRE holds here too for our proposed method, and show that our method performs again much better than the PB method. For the PB method, the relative error seems to increase, but it stabilizes for smaller values of ϵ.

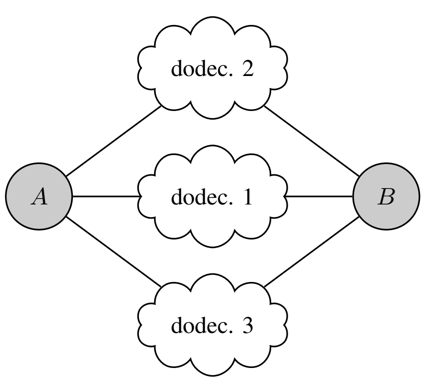

Fig. 7. Three dodecahedron topologies in parallel.

Example 10: We consider a similar construction as in the previous example, except that the three dodecaheron copies are connected in series instead of in parallel, as shown in Fig. 8. The source node is the source of the first copy, and the destination is the destination of the third one, while the destination of the first (respectively second) copy is the source of the second (respectively third). We still consider the homogeneous case, where all links have the same unreliability ϵ. Here the reliability (and not the unreliability) is the cube of that of a single dodecahedron topology. The empirical results in Table VIII suggest that only BRE holds, but with a smaller variance than the PB method.

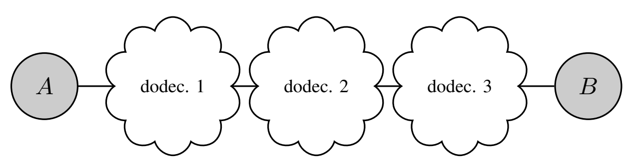

Fig. 8. Three dodecahedron topologies in series.

The gain in accuracy (variance reduction) with the proposed method comes at the expense of increased computing times, because we need to find two mincuts with maximal probability at each step of the sampling process. This search is done in polynomial time, using for instance the basic Ford-Fulkerson algorithm [29]. To give some idea of the added computational burden compared with crude Monte Carlo, we report here the average CPU time to simulate one realization of the graph with both methods, with our Java implementation on a MacBook with a 2.4 GHz Intel Core 2 Duo processor. For Example 4, the average simulation time is 9.4×10^{-6} seconds with crude

TABLE VII
EMPIRICAL RESULTS FOR EXAMPLE 9, FOR $n = 10^4$, AND THREE VALUES OF ϵ.

ϵ	Estimate	95% confidence interval	STD	RE	STD-PB	RE-PB
10^{-1}	2.3573×10^{-8}	$(2.2496 \times 10^{-8},\ 2.4650 \times 10^{-8})$	5.49×10^{-8}	2.3	2.95054228E-6	125
5×10^{-2}	2.5732×10^{-11}	$(2.5138 \times 10^{-11},\ 2.6327 \times 10^{-11})$	3.03×10^{-11}	1.2	7.10997524E-9	280
10^{-2}	8.7655×10^{-18}	$(8.7145 \times 10^{-18},\ 8.8165 \times 10^{-18})$	2.60×10^{-18}	0.30	5.15889144E-15	314

TABLE VIII
EMPIRICAL RESULTS FOR EXAMPLE 10, FOR $n = 10^4$, AND THREE VALUES OF ϵ.

ϵ	Estimate	95% confidence interval	STD	RE	STD-PB	RE-PB
10^{-2}	6.1919×10^{-6}	$(6.0190 \times 10^{-6},\ 6.3648 \times 10^{-6})$	8.82×10^{-6}	1.4	1.369×10^{-4}	22.1
10^{-3}	5.9730×10^{-9}	$(5.8213 \times 10^{-9},\ 6.1146 \times 10^{-9})$	7.74×10^{-9}	1.3	1.477×10^{-7}	24.7
10^{-4}	5.9246×10^{-12}	$(5.7766 \times 10^{-12},\ 6.0727 \times 10^{-12})$	7.55×10^{-12}	1.3	1.578×10^{-10}	26.6

Monte Carlo, and 4.2×10^{-5} seconds with our mincut-maxprob IS algorithm. For Example 8 (the dodecahedron topology), the numbers are 4.8×10^{-5} seconds for crude Monte Carlo, and 7.9×10^{-4} seconds for our method, so the time increases by a factor 16.5. For Example 9, this factor is about 52 while it is 26 for Example 10. These factors are not negligible, but they are modest in comparison with the (arbitrarily large) factors by which the variance can be reduced.

VII. CONCLUSIONS

We have described in this paper a new importance sampling scheme for the simulation of static reliability models. Our method tries to mimic the zero-variance change of measure, sampling sequentially the links, and approaching the unreliability of the model given the status of previously sampled links by a rough estimation, here taken as the probability of a mincut. We proved that the BRE property is verified by our scheme when the unreliabilities of all links go to zero. It is therefore possible to construct examples where the variance can be reduced by an arbitrary large factor with respect to MC, by taking ϵ small enough. Under additional conditions on the quality of the approximation, even VRE is verified. The efficiency of the method is illustrated on small examples to check the correctness of the implementation, and on a larger model often used in the area for illustration purposes. The VRE property is observed on benchmark examples, something not observed by the other rare-event simulation methods of the literature, emphasizing the interest of our proposal when links are very reliable.

This work will be pursued in several directions. We will first try to improve the approximation of the unreliability function, likely driving to a change of measure closer to the zero-variance one. The idea is to consider not only the mincut probability function, which is a lower bound of the unreliability, but other functions such as an upper-bound taken as one minus the probability of a minpath, and to use a linear combination of those *basis functions* to get a better approximation. This is similar to what is done in dynamic programming to approximate the Bellman value function. The coefficients of the linear combination will have to be learned. Another issue we would like to tackle is the combination of this importance sampling scheme with other variance reduction techniques such as those in [6], [8], where importance sampling is not considered. Adding an adapted zero-variance IS approximation to the techniques proposed there is likely to drastically reduce the variance, and produce an extremely efficient estimator. A third direction for improvement is to combine our general procedure with graph reduction techniques that can be applied in polynomial time as a function of the model size, usually employed in the area, such as series-parallel reductions for example. These reductions should contribute to further diminish both the variance and the global cost in time of the method.

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