Approximation of Zero-Variance Importance Sampling in Reliability Settings

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Outline

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Introduction: rare events

Rare events occur when dealing with performance evaluation in many different areas

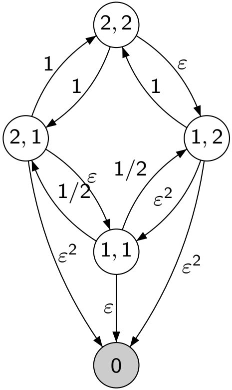
- in *telecommunication networks*: loss probability of a small unit of information (a packet, or a cell in ATM networks), connectivity of a set of nodes,
- in *dependability analysis*: probability that a system is failed at a given time, availability, mean-time-to-failure,
- in *air control systems*: probability of collision of two aircrafts,
- in *particle transport*: probability of penetration of a nuclear shield,
- in *biology*: probability of some molecular reactions,
- in *insurance*: probability of ruin of a company,
- in *finance*: value at risk (maximal loss with a given probability in a predefined time),
- ...

What is a rare event? Why simulation?

- A rare event is an event occurring with a small probability.
- How small? Depends on the context.
- In many cases, these probabilities can be between 10^{-8} and 10^{-10}, or even at lower values. Main example: critical systems, that is,
  - systems where the rare event is a catastrophic failure with possible human losses,
  - or systems where the rare event is a catastrophic failure with possible monetary losses.
- In most of the above problems, the mathematical model is often too complicated to be solved by analytic or numeic methods because
  - the assumptions are not stringent enough,
  - the mathematical dimension of the problem is too large,
  - the state space is too large to get a result in reasonable time,
  - ...
- Simulation is, most of the time, the only tool at hand.

Example: Highly Reliable Markovian Systems (HRMS)

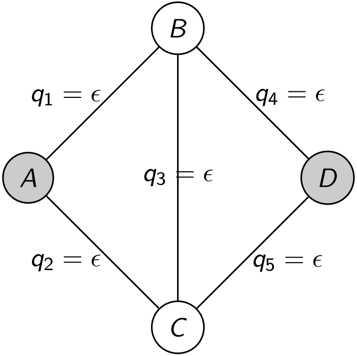
- System with c types of components. $Y = (Y_1, \ldots, Y_c)$ with Y_i number of up components.
- $\mathbf{1}$: state with all components up.
- Markov chain. Failure rates are $O(\varepsilon)$, but not repair rates. Failure propagations possible.
- System down when in grey state(s).
- Goal: compute $\mu(y)$ probability to hit Δ before $\mathbf{1}$.
- $\mu(\mathbf{1})$ important in dependability analysis,
- Small if ε small.



Example: connectivity within a graph

- *Static* reliability problems (*time* is not an explicit variable)
- Communication network:
  - nodes assumed to be perfect,
  - links can fail independently.
  - For each edge e, *elementary unreliability* q_e, reliability $r_e = 1 - q_e$.
  - The network works iff two specific nodes communicate.
- Model: graph with M links
- $X = (X_1, \ldots, X_M)$ (random) *configuration* with $X_e = 1$ if edge e works, 0 otherwise.
- state of the system: $\phi(X)$, where $\phi(X) = 1$ iff s and t not connected.

- $u = \mathbb{E}[\phi(X)]$, computation NP-hard problem in general.
- u small if individual unreliabilities small and/or redundancy of paths.



Outline

Monte Carlo

- In all the above problems, the goal is to compute $\mu = \mathbb{E}[X]$ of some random variable X.
- Monte Carlo simulation (in its basic form) generates n independent copies of X, $(X_i, 1 \leq i \leq n)$,
- $\bar{X}_n = (1/n)\sum_{i=1}^{n} X_i$ approximation of μ.
- Almost sure convergence as $n \to \infty$ (law of large numbers).
- Accuracy: central limit theorem, yielding a confidence interval

$$\mu \in \left(\bar{X}_n - \frac{c_\alpha \sigma}{\sqrt{n}}, \bar{X}_n + \frac{c_\alpha \sigma}{\sqrt{n}}\right)$$

  - α: desired confidence probability,
  - $c_\alpha = \Phi^{-1}(1 - \frac{\alpha}{2})$ with Φ is the cumulative Normal distribution function of $\mathcal{N}(0,1)$
  - $\sigma^2 = \mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$, estimated by $S_n^2 = (1/(n-1))\sum_{i=1}^{n} X_i^2 - (n/(n-1))(\bar{X}_n)^2$.

Inefficiency of crude Monte Carlo

- *Crude* Monte Carlo: simulates the model directly
- We compute the probability $\mu = \mathbb{E}[1_A] << 1$ of a rare event A.
- X_i Bernoulli r.v.: 1 if the event is hit and 0 otherwise.
- To get a single occurence, we need in average $1/\mu$ replications (10^9 for $\mu = 10^{-9}$), and more to get a confidence interval.
- $n\bar{X}_n$ Binomial with parameters (n, μ) and the confidence interval is

$$\left(\bar{X}_n - \frac{c_\alpha\sqrt{\mu(1-\mu)}}{\sqrt{n}}, \bar{X}_n + \frac{c_\alpha\sqrt{\mu(1-\mu)}}{\sqrt{n}}\right).$$

- *Relative half width* $c_\alpha\sigma/(\sqrt{n}\mu) = c_\alpha\sqrt{(1-\mu)/\mu/n} \to \infty$ as $\mu \to 0$.
- For a given relative error RE , the required value of

$$n = (c_\alpha)^2\frac{1-\mu}{RE^2\mu},$$

  inversely proportional to μ.
- Two main families of techniques:
  - Splitting
  - Importance Sampling

Robustness properties

- In rare-event simulation models, we often parameterize with a rarity parameter $\epsilon > 0$ such that $\mu = \mathbb{E}[X(\epsilon)] \to 0$ as $\epsilon \to 0$.
- An estimator $X(\epsilon)$ is said to have *bounded relative variance* (or *bounded relative error*) if $\sigma^2(X(\epsilon))/\mu^2(\epsilon)$ is bounded uniformly in ϵ.
- Interpretation: estimating $\mu(\epsilon)$ with a given relative accuracy can be achieved with a bounded number of replications even if $\epsilon \to 0$.
- Weaker property: *asymptotic optimality* (or *logarithmic efficiency*) if $\lim_{\epsilon \to 0} \ln(\mathbb{E}[X^2(\epsilon)])/\ln(\mu(\epsilon)) = 2$.
- Stronger property: *vanishing relative variance*: $\sigma^2(X(\epsilon))/\mu^2(\epsilon) \to 0$ as $\epsilon \to 0$. Asymptotically, we get the zero-variance estimator.
- Other robustness measures exist (based on higher degree moments, on the Normal approximation, on simulation time...).

L'Ecuyer, Blanchet, T., Glynn, ACM ToMaCS 2010

Outline

Importance Sampling (IS)

- Let $X = h(Y)$ for some function h where Y obeys some probability law $\mathbb{P}$.
- IS replaces $\mathbb{P}$ by another probability measure $\tilde{\mathbb{P}}$, using

$$E[X] = \int h(y) d\mathbb{P}(y) = \int h(y) \frac{d\mathbb{P}(y)}{d\tilde{\mathbb{P}}(y)} d\tilde{\mathbb{P}}(y) = \tilde{\mathbb{E}}\left[h(Y)L(Y)\right]$$

  - $L = d\mathbb{P}/d\tilde{\mathbb{P}}$ likelihood ratio,
  - $\tilde{\mathbb{E}}$ is the expectation associated to probability law $\mathbb{P}$.
- Required condition: $d\tilde{\mathbb{P}}(y) \neq 0$ when $h(y)d\mathbb{P}(y) \neq 0$.
- If $\mathbb{P}$ and $\tilde{\mathbb{P}}$ continuous laws, L ratio of density functions.
- If $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are discrete laws, L ratio of indiv. prob.
- Unbiased estimator: $\displaystyle\frac{1}{n}\sum_{i=1}^{n} h(Y_i)L(Y_i)$ with $(Y_i,\, 1 \leq i \leq n)$ i.i.d; copies of Y, according to $\tilde{\mathbb{P}}$.
- Goal: select probability law $\tilde{\mathbb{P}}$ such that

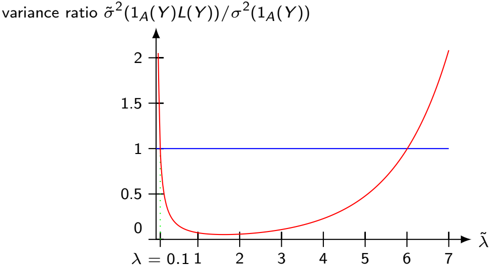
$$\tilde{\sigma}^2[h(Y)L(Y)] = \tilde{\mathbb{E}}[(h(Y)L(Y))^2] - \mu^2 < \sigma^2[h(Y)].$$

IS difficulty: system with exponential failure time

- Goal: to compute μ that the system fails before T, $\mu = \mathbb{E}[1_A(Y)] = 1 - e^{-\lambda T}$.
- Use for IS an exponential density with a different rate $\tilde{\lambda}$

$$\tilde{\mathbb{E}}[(1_A(Y)L(Y))^2] = \int_0^T \left(\frac{\lambda e^{-\lambda y}}{\tilde{\lambda} e^{-\tilde{\lambda} y}}\right)^2 \tilde{\lambda} e^{-\tilde{\lambda} y} dy = \frac{\lambda^2(1 - e^{-(2\lambda - \tilde{\lambda})T})}{\tilde{\lambda}(2\lambda - \tilde{\lambda})}.$$

- Variance ratio for $T = 1$ and $\lambda = 0.1$:



Optimal estimator for estimating $\mathbb{E}[h(Y)] = \int h(y)L(y)d\tilde{\mathbb{P}}(y)$

- Optimal change of measure:

$$d\tilde{\mathbb{P}} = \frac{|h(Y)|}{\mathbb{E}[|h(Y)|]} d\mathbb{P}.$$

- *Proof:* for any alternative IS measure $\mathbb{P}'$, leading to the likelihood ratio L' and expectation $\mathbb{E}'$,

$$\tilde{\mathbb{E}}[(h(Y)L(Y))^2] = (\mathbb{E}[|h(Y)|])^2 = (\mathbb{E}'[|h(Y)|L'(Y)])^2 \le \mathbb{E}'[(h(Y)L'(Y))^2].$$

- If $h \ge 0$, $\tilde{\mathbb{E}}[(h(Y)L(Y))^2] = (\mathbb{E}[h(Y)])^2$, i.e., $\tilde{\sigma}^2(h(Y)L(Y)) = 0$. That is, IS provides a zero-variance estimator.
- Implementing it requires knowing $\mathbb{E}[|h(Y)|]$, i.e. what we want to compute; if so, no need to simulation!
- But provides a hint on the general form of a “good” IS. measure.

IS for a discrete-time Markov chain (DTMC) $\{Y_j, j \geq 0\}$

- $X = h(Y_0, \ldots, Y_\tau)$ function of the sample path with
  - $P = (P(y, z)$ transition matrix, $\pi_0(y) = \mathbb{P}[Y_0 = y]$, initial probabilities
  - up to a stopping time τ, first time it hits a set Δ.
  - $\mu(y) = \mathbb{E}_y[X]$.
- IS replaces the probabilities of paths $(y_0, \ldots, y_n)$,

$$\mathbb{P}[(Y_0, \ldots, Y_\tau) = (y_0, \ldots, y_n)] = \pi_0(y_0) \prod_{j=1}^{n-1} P(y_{j-1}, y_j),$$

  by $\tilde{\mathbb{P}}[(Y_0, \ldots, Y_\tau) = (y_0, \ldots, y_n)]$ st $\tilde{\mathbb{E}}[\tau] < \infty$.
- For convenience, the IS measure remains a DTMC, replacing $P(y, z)$ by $\tilde{P}(y, z)$ and $\pi_0(y)$ by $\tilde{\pi}_0(y)$.
- Then $L(Y_0, \ldots, Y_\tau) = \dfrac{\pi_0(Y_0)}{\tilde{\pi}_0(Y_0)} \displaystyle\prod_{j=1}^{\tau-1} \dfrac{P(Y_{j-1}, Y_j)}{\tilde{P}(Y_{j-1}, Y_j)}$.

Zero-variance IS estimator for Markov chains simulation

- Restrict to an additive (positive) cost

$$X = \sum_{j=1}^{\tau} c(Y_{j-1}, Y_j)$$

- Is there a Markov chain change of measure yielding zero-variance?
- Yes we have zero variance with

$$\begin{aligned}\tilde{P}(y,z) &= \frac{P(y,z)(c(y,z)+\mu(z))}{\sum_w P(y,w)(c(y,w)+\mu(w))} \\ &= \frac{P(y,z)(c(y,z)+\mu(z))}{\mu(y)}.\end{aligned}$$

- Without the additivity assumption the probabilities for the next state must depend in general of the entire history of the chain.

Zero-variance for Markov chains

- Proof by induction on the value taken by τ, using the fact that $\mu(Y_\tau) = 0$ In that case, if $\tilde{X}$ denotes the IS estimator,

$$
\begin{aligned}
\tilde{X} &= \sum_{i=1}^{\tau} c(Y_{i-1}, Y_i) \prod_{j=1}^{i} \frac{P(Y_{j-1}, Y_j)}{\tilde{P}(Y_{j-1}, Y_j)} \\
&= \sum_{i=1}^{\tau} c(Y_{i-1}, Y_i) \prod_{j=1}^{i} \frac{P(Y_{j-1}, Y_j)\mu(Y_{j-1})}{P(Y_{j-1}, Y_j)(c(Y_{j-1}, Y_j) + \mu(Y_j))} \\
&= \sum_{i=1}^{\tau} c(Y_{i-1}, Y_i) \prod_{j=1}^{i} \frac{\mu(Y_{j-1})}{c(Y_{j-1}, Y_j) + \mu(Y_j)} \\
&= \mu(Y_0)
\end{aligned}
$$

- *Unique* Markov chain implementation of the zero-variance estimator.
- Again, implementing it requires knowing $\mu(y)\ \forall y$, the quantities we wish to compute.
- Approximation to be used.

Zero-variance approximation

- Use a heuristic approximation $\hat{\mu}(\cdot)$ and plug it into the zero-variance change of measure instead of $\mu(\cdot)$.
- More efficient but also more requiring technique: *learn adaptively* function $\mu(\cdot)$, and still plug the approximation into the zero-variance change of measure formula instead of $\mu(\cdot)$.
  - *Adaptive Monte Carlo* (AMC) proceeds iteratively.
    - Considers several steps and n_i independent simulation replications at step i.
    - At step i, replaces $\mu(x)$ by a guess $\mu^{(i)}(x)$
    - use probabilities $\tilde{P}^{(i)}_{y,z} = \dfrac{P_{y,z}(c_{y,z} + \mu^{(i)}(z))}{\sum_w P_{y,w}(c_{y,w} + \mu^{(i)}(w))}$.
    - Gives a new estimation $\mu^{(i+1)}(y)$ of $\mu(y)$, from which a new transition matrix $\tilde{P}^{(i+1)}$ is defined.
  - *Adaptive stochastic approximation* (ASA) updates the probabilities at each step of the simulation.
  - But those methods require to store a lot of information for large systems.
- Other methods, based on subsolutions of Isaac equations (P. Dupuis *et al.*) or the construction of Lyapounov functions (Blanchet, Glynn *et al.*).

Illustration of heuristics: birth-death process

- Let $P(i, i+1) = p$ and $P(i, i-1) = 1 - p$ for $1 \leq i \leq B - 1$, and $P(0, 1) = P(B, B-1) = 1$.
- We want to compute $\mu(1)$, probability of reaching B before coming back to 0.
- If p small, to approach $\mu(\cdot)$, we can use

$$\hat{\mu}(y) = p^{B-y} \quad \forall y \in \{1, \ldots, B-1\}$$

  with $\hat{\mu}(0) = 0$ and $\hat{\mu}(B) = 1$ based on the asymptotic estimate $\mu(i) = p^{B-i} + o(p^{B-i})$.
- We can verify that the variance of this estimator is going to 0 (for fixed sample size) as $p \to 0$.

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- System with c types of components. $Y = (Y_1, \ldots, Y_c)$ with Y_i number of up components.
- $\mathbf{1}$: state with all components up.
- Failure rates are $O(\varepsilon)$, but not repair rates. Failure propagations possible.
- System down (in Δ) when some combinations of components are down.
- Goal: compute $\mu(\mathbf{1})$ with $\mu(y)$ probability to hit Δ before $\mathbf{1}$.
- Simulation using the embedded DTMC. Failure probabilities are $O(\varepsilon)$ (except from $\mathbf{1}$). How to improve (accelerate) this?
- Existing method: $\forall y \neq \mathbf{1}$, increase the probability of the set of failures to constant $0.5 < q < 0.9$ and use individual probabilities proportional to the original ones (SFB), or uniformly (BFB).
- Failures not rare anymore. BRE property verified for BFB.

HRMS Example, and IS

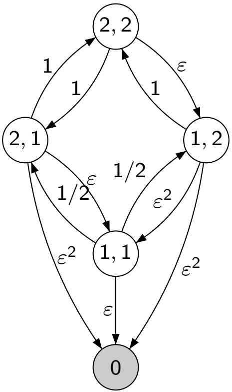

Figure: Original probabilities

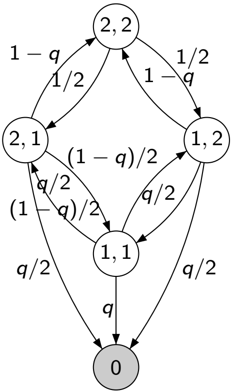

Figure: Probabilities under IS/BFB

HRMS, Zero-variance IS

L'Ecuyer & T., ANOR, to appear

- Complicates the previous model due to the multidimensional description of a state.
- The idea is to approach $\mu(y)$ by the probability of the path from y to Δ with the largest probability
- Intuition: as $\epsilon \to 0$, we get a good idea of the probability.

Proposition

Bounded Relative Error proved (as $\epsilon \to 0$) in general.
Even Vanishing Relative Error if $\hat{\mu}(y)$ contains all the paths with the smallest degree in ϵ.

- Other simple version: approach $\mu(y)$ by the (sum of) probability of paths from y with only failure components of a given type.
- Gain of several orders of magnitudes + stability of the results with respect to the literature.

HRMS: numerical illustrations

- Comparison of BFB and Zero-Variance Approximation (ZVA).
- $c = 3$ types of components, n_i of type i
- $\lambda_1 = \varepsilon$, $\lambda_2 = 1.5\varepsilon$, and $\lambda_3 = 2\varepsilon^2$, $\mu = 1$
- System is down whenever fewer than two components of any one type are operational.

n_i	ε	μ_0	BFB est	ZVA est	BFB σ^2	ZVA σ^2
3	0.001	2.6×10^{-3}	2.7×10^{-3}	2.6×10^{-3}	6.2×10^{-5}	2.2×10^{-8}
6	0.01	1.8×10^{-7}	1.9×10^{-7}	1.8×10^{-7}	6.3×10^{-11}	2.0×10^{-14}
6	0.001	1.7×10^{-11}	1.8×10^{-11}	1.7×10^{-11}	8.8×10^{-19}	1.2×10^{-23}
12	0.1	6.0×10^{-8}	4.8×10^{-8}	6.0×10^{-8}	8.1×10^{-10}	1.6×10^{-10}
12	0.001	3.9×10^{-28}	(1.8×10^{-40})	3.9×10^{-28}	(3.2×10^{-74})	1.4×10^{-55}

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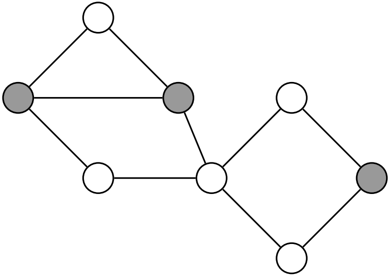
Graph model

- M links can fail independently, *elementary unreliability* $q_e = 1 - r_e$ for edge e.
- What is the probability that the set $\mathcal{K}$ of (grey) nodes is connected (in the underlying random partial graph of $\mathcal{G}$)?
- $X = (X_1, \ldots, X_M)$ (random) *configuration* with $X_e = 1$ if edge e works, 0 otherwise.
- state of the system: $\phi(X)$, where $\phi(X) = 1$ iff $\mathcal{K}$ not connected.
- $u = \mathbb{E}[\phi(X)] = \sum_{x \in \{0,1\}^M} \phi(x)\mathbb{P}[X = x]$.



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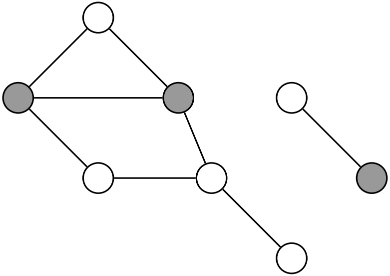

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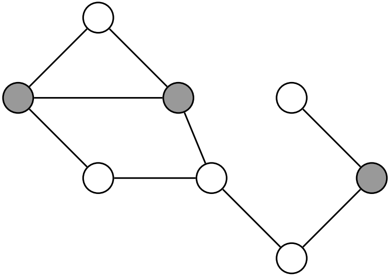

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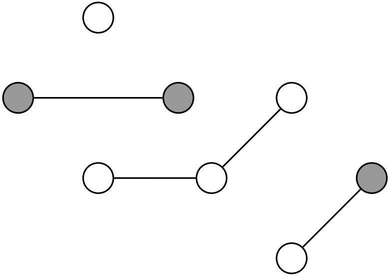

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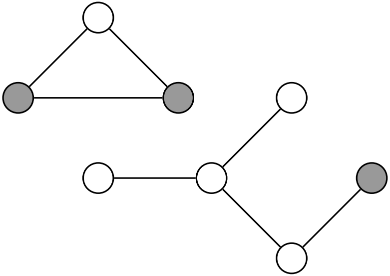

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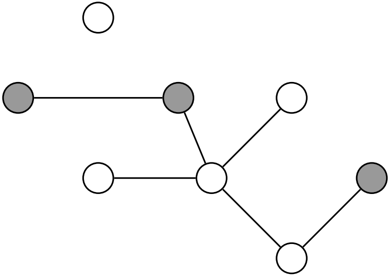

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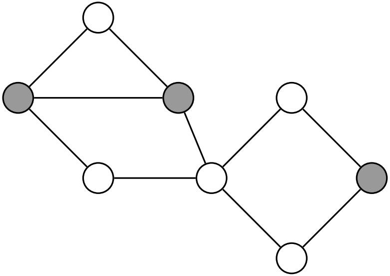


- We have to sum over the 2^M configurations.

Zero-variance est. L'Ecuyer, Rubino, Saggadi & T., IEEE Trans. Rel. 2011

- Idea: sample the links one after the other, with an IS probability that *depends on the state of previously sampled links*.
- Let $u_m(x_1, \cdots, x_{m-1})$, with $x_i \in \{0, 1\}$, be the unreliability of the graph G given the states of the links 1 to $m-1$: if $x_i = 1$ the link i is operational, otherwise it is failed.
- Then $u = u_1()$.
- Sample state of link m, giving 1 with probability:

$$r'_m(x_1, \cdots, x_{m-1}) = \frac{r_m u_{m+1}(x_1, \cdots, x_{m-1}, 1)}{r_m u_{m+1}(x_1, \cdots, x_{m-1}, 1) + (1 - r_m) u_{m+1}(x_1, \cdots, x_{m-1}, 0)}.$$

- Remark (by conditionning) that

$$u_m(x_1, \cdots, x_{m-1}) = r_m u_{m+1}(x_1, \cdots, x_{m-1}, 1) + (1 - r_m) r_m u_{m+1}(x_1, \cdots, x_{m-1}, 0).$$

Zero-variance estimation and approximation

Proposition

Using this IS, the estimator has zero variance (always yields u).

- Problem: the $u_m(\cdot)$ are not known, otherwise no need to simulate.
- Principle: approach $u_m(\cdot)$ by some $\hat{u}_m(\cdot)$ and use

$$r'_m(x_1, \cdots, x_{m-1}) = \frac{r_m \hat{u}_{m+1}(x_1, \cdots, x_{m-1}, 1)}{r_m \hat{u}_{m+1}(x_1, \cdots, x_{m-1}, 1) + (1 - r_m)\hat{u}_{m+1}(x_1, \cdots, x_{m-1}, 0)}.$$

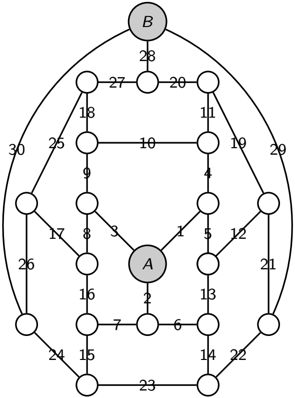
Approximation of the zero-variance estimator

- Our proposal: $\hat{u}_m(x_1, \cdots, x_{m-1})$ is the probability of a mincut of the graph with highest probability, given the state of links 1 to $m-1$.
  - A *cut* (or $\mathcal{K}$-cut) is a set of edges such that, if we remove them, the nodes in $\mathcal{K}$ are not in the same connected component.
  - A *mincut* (minimal cut) is a cut that contains no other cut than itself.
- Intuition: the unreliability is the probability of union of all cuts, the most crucial one(s) being the mincut(s) with highest probability.
- Cuts can be obtained in polynomial time.
- In a given state $(x_1, \cdots, x_{m-1})$, we need to determine $\hat{u}_{m+1}(x_1, \cdots, x_{m-1}, 1)$ and $\hat{u}_{m+1}(x_1, \cdots, x_{m-1}, 0)$.
- This adds some computational burden, but should substantially reduce the variance.

Proposition

Bounded relative error proved in general,
Vanishing relative error under identified conditions.

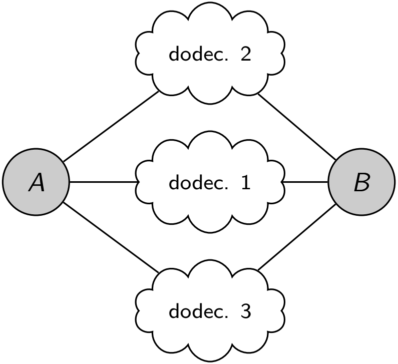
Ex: dodecahedron topology, all links with unreliability ϵ



ϵ	Estimation	Confidence interval	Std deviation	Relative error
10^{-1}	$2.8960\ 10^{-3}$	$(2.8276\ 10^{-3}, 2.9645\ 10^{-3})$	$3.49\ 10^{-3}$	1.2
10^{-2}	$2.0678\ 10^{-6}$	$(2.0611\ 10^{-6}, 2.0744\ 10^{-6})$	$3.42\ 10^{-7}$	0.17
10^{-3}	$2.0076\ 10^{-9}$	$(2.0053\ 10^{-9}, 2.0099\ 10^{-9})$	$1.14\ 10^{-10}$	0.057
10^{-4}	2.0007^{-12}	$(2.0000\ 10^{-12}, 2.0014\ 10^{-12})$	$3.46\ 10^{-14}$	0.017

With respect to crude MC, a computational time increase of 16.

Larger networks: 3 dodecahedrons in parallel

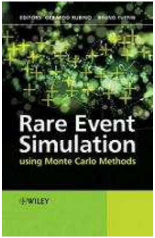


ϵ	Estimate	95% confidence interval	std dev.	Relative Error
10^{-1}	2.3573×10^{-8}	$(2.2496 \times 10^{-8},\ 2.4650 \times 10^{-8})$	5.49×10^{-8}	2.3
5×10^{-2}	2.5732×10^{-11}	$(2.5138 \times 10^{-11},\ 2.6327 \times 10^{-11})$	3.03×10^{-11}	1.2
10^{-2}	8.7655×10^{-18}	$(8.7145 \times 10^{-18},\ 8.8165 \times 10^{-18})$	2.60×10^{-18}	0.30

- Vanishing relative error observed
- For 3 dodecahedron in series, Bounded relative error observed
- Works very well for such topologies with close to 100 links, and larger.

Conclusions: book advertisements

Released in March 2009 (John Wiley & Sons):



In March 2010 (éditions Hermès):