Evolutionary Games in Wireless Networks

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***Abstract*—We consider a noncooperative interaction among a large population of mobiles that interfere with each other through many local interactions. The first objective of this paper is to extend the evolutionary game framework to allow an arbitrary number of mobiles that are involved in a local interaction. We allow for interactions between mobiles that are not necessarily *reciprocal*. We study 1) multiple-access control in a slotted Aloha-based wireless network and 2) power control in wideband code-division multiple-access wireless networks. We define and characterize the equilibrium (called evolutionarily stable strategy) for these games and study the influence of wireless channels and pricing on the evolution of dynamics and the equilibrium.**

***Index Terms*—Evolutionarily stable strategy (ESS), evolutionary game, slotted Aloha, wideband code-division multiple access (W-CDMA).**

I. Introduction

EVOLUTIONARY game formalism is a central mathematical tool developed by biologists for predicting population dynamics in the context of *interactions between populations* through pairwise interactions. This formalism studies evolutionary stability and *evolutionary game dynamics*. The evolutionarily stable strategy (ESS), first defined in [5], is characterized by robustness against invaders (mutations): 1) if an ESS is reached, then the proportions of each population do not change in time, and 2) at ESS, the populations are immune from being invaded by other small populations. This notion is stronger than Nash equilibrium in which it is only requested that a single user would not benefit by a change (mutation) of its behavior. Although ESS has been defined in the context of biological systems, it is highly relevant to engineering as well [13]. In the biological context, replicator dynamics is a model that is used to explain observed variations in a population size. In engineering, we can go beyond characterizing and modeling existing evolution. The evolution of protocols can be engineered by providing guidelines or regulations for the way to upgrade existing ones and in determining parameters that are related to deployment of new protocols and services. There has been much work on noncooperative modeling of power control and multiple-access control using game theory. There are two advantages in doing so within the framework of evolutionary games: 1) It provides the stronger concept of equilibria, the ESS, which allows us to identify robustness against deviations of more than one mobile, and 2) it allows us to apply the generic convergence theory of evolutionary game dynamics and stability results that we will introduce in future sections.

Contributions: In [7]–[9], we have studied evolutionary games with *pairwise local interactions* in the context of wireless networks. Our first contribution is to extend the evolutionary game framework to allow an arbitrary (possibly random) number of players that are involved in local interactions (possibly nonreciprocal). We then apply the extended model to multiple-access control games with more than two interacting nodes. We study the impact of the node distribution in the game area on equilibrium stable strategies. We use the notion of expected utility, as this game is not symmetric. We analyze the impact of the cost components (transmission cost, collision cost, and regret cost) on the probability of successful transmission and present some optimization issues. Our third contribution is to apply evolutionary game models to study the interaction of numerous mobiles in competition in a wideband code-division multiple-access (W-CDMA) wireless environment.

Organization of This Paper: Section II provides motivating examples that illustrate the limitation of current evolutionary games. Section III studies our general model of population games with a random number of interacting players at each local interaction. We study in Sections IV and V the generalized multiple-access game and the evolutionary uplink power control in W-CDMA networks. Section VI presents numerical examples on the impact of time delays, and Section VII concludes this paper.

II. Examples: Symmetry and Reciprocity

Assume that each individual in a large population occasionally needs to take some action. When doing so, it interacts with the actions of some K (possibly a random number of) other individuals. We shall consider throughout this paper symmetric games in the sense that any individual faces the same type of game. All players have the same actions available and same *expected* utility. We note, however, that the actual realizations need not be symmetric. In particular, 1) the number of players with which a given player interacts may vary from one player to another, and 2) we do not even need the reciprocity property: If player A interacts with player B, we do not require the converse to hold. We provide some examples of multiple-access games to illustrate this nonreciprocity. For example, consider local

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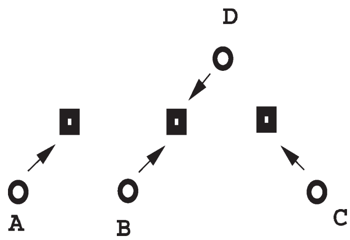

Fig. 1. Nonreciprocal pairwise interactions.

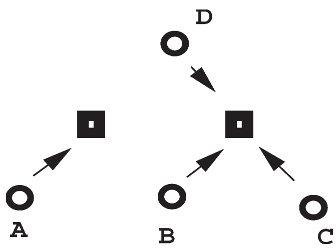

Fig. 2. Nonreciprocal interactions between groups of three players.

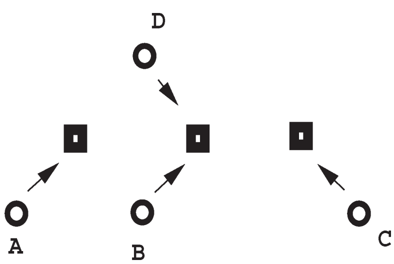

Fig. 3. Interactions between a random number of players.

interactions between transmitters; for each transmitter, there corresponds a receiver. We shall say that transmitter A is subject to an interaction (interference) from transmitter B if the receiver of the transmission from A is within the interference range of transmitter B.

Nonreciprocal Pairwise Interactions: Consider the example depicted in Fig. 1. It contains four sources (circles) and three destinations (squares). A transmission of a source i within a distance r of the receiver R causes interference to a transmission from a source $j \neq i$ to receiver R. We see that source a and source c cause no interference to any other transmission, but the transmission from A suffers from the interference from source B, and the one from C suffers from the transmission of the topmost source (called D). Sources B and D interfere with each other at their common destination. Thus, each of the four sources suffers interference from a single other source; however, except for nodes B and D, the interference is not reciprocal.

Nonreciprocal Nonpairwise Interactions: In Fig. 2, there are four sources and only two destinations. Node A does not cause any interference to the other nodes but suffers interference from nodes B and D. Nodes B, C, and D interfere with each other. This is a situation in which each mobile is involved in interference from two other mobiles; however, again, the interference is not reciprocal.

Interactions Between a Random Number of Players: In this example, the number of interfering nodes is not fixed (Fig. 3). Node A suffers interference from two nodes, nodes B and D suffer interference from a single other node, and node C does not suffer (and does not cause) interference. Note that if D or B moves from its position, the mobile A can play the role of C in this analysis, and a new game between B or D and C will be played.

All examples exhibit asymmetric realizations and nonreciprocity. We next show how such a situation can still be considered as symmetric (due to the fact that we consider distributions of nodes rather than realizations). Assume that the location of the transmitters follows a Poisson distribution with parameter λ over the 2-D plane. Consider an arbitrary user A. Let r be the interference range. Then, the number of transmitters within the interference range of the receiver of A has a Poisson distribution with parameter $\lambda\pi r^2/2$. Since this holds for any node, the game is considered to be symmetric. The reason that the distribution is taken into account rather than the realization is that we shall assume that the actions of players will be taken before knowing the realization.

III. Extending Evolutionary Games

We extend the evolutionary game framework to allow an arbitrary (possibly random) number of players that are involved in a local interaction. First, we present the model and notations.

– There is one population of users. The number of users in the population is large.
– We assume that there are finitely many pure strategies or actions. Each member of the population chooses from the same set of strategies $\mathcal{A} = \{1, 2, \ldots, N\}$.
– Let $M := \{(x_1, \ldots, x_N) | x_j \geq 0, \sum_{j=1}^{N} x_j = 1\}$ be the set of probability distributions over the N pure actions. M can be interpreted as the set of mixed strategies. It is also interpreted as the set of distributions of strategies among the population, where x_j represents the proportion of users choosing strategy j. A distribution x is sometimes called the "state" or the "profile" of the population.
– The number of users interfering with a given randomly selected user is a random variable K in the set $\{0, 1, \ldots\}$. In the bounded case, we will denote by $k_{\max}$ the maximum number of interacting users simultaneously with a user. This value depends on the node density and the transmission range. When making a choice of a strategy, a player knows the distribution of K but not its realization.
– A player does not know how many players would interact with it.
– The payoff function of all players depends on the player's own behavior and the behavior of the other players. The expected payoff of a user playing strategy j when the state of the population is x is given by $f(j, x) = \sum_{k \geq 0} \mathbb{P}(K = k) u_k(j, x)$, where $u_k(j, x)$ is the payoff of a user playing strategy j when the state of the population is x and given that the number of users interfering with a given randomly selected user is k. Although the payoffs are symmetric, the actual interference or interactions between two players that use

the same strategy need not be the same, allowing for nonreciprocal behavior. The reason is that the latter is a property of the random realization, whereas the actual payoff already averages over the randomness related to the interactions, the number of interfering players, the topology, etc.

– The game is played many times.

Solution Concepts and Refinement: Suppose that, initially, the population profile is $x \in M$. The average payoff is $f(x,x) := \sum_{j=1}^{N} x_j f(j,x)$. Now, suppose that a small group of mutants enters the population playing according to a different profile $mut \in M$. If we call $\epsilon \in (0,1)$ the size of the subpopulation of mutants after normalization, then the population profile after mutation will be $\epsilon mut + (1-\epsilon)x$. After mutation, the average payoff of nonmutants will be given by $f(x, \epsilon mut + (1-\epsilon)x)$, where $f(x,y) := \sum_{j=1}^{N} x_j f(j,y)$. Note that f need not be linear in the second variable. Analogously, the average payoff of a mutant is $f(mut, \epsilon mut + (1-\epsilon)x)$.

ESS (or State): A strategy $x \in M$ is an ESS if for any $mut \neq x$, there exists some $\epsilon_{mut} \in (0,1)$, which may depend on mut, such that for all $\epsilon \in (0, \epsilon_{mut})$, one has

$$f\left(x, \epsilon mut + (1-\epsilon)x\right) > f\left(mut, \epsilon mut + (1-\epsilon)x\right) \quad (1)$$

which can be rewritten as $\sum_{j=1}^{N}(x_j - mut_j) f(j, \epsilon mut + (1-\epsilon)x) > 0$. That is, x is an ESS if, after mutation, nonmutants are more successful than mutants. In other words, mutants cannot invade the population and will eventually get extinct.

Neutrally Stable Strategy: We say that x is a neutrally stable strategy if inequality (1) is nonstrict. A population profile x is an equilibrium if the variational inequality

$$\sum_{j \in \mathcal{A}} (x_j - mut_j) f(j,x) \geq 0$$

holds for all $mut \in M$. Note that an ESS is a neutrally stable strategy that is a symmetric equilibrium of the one-shot game, i.e., the strategy is the same for each player, and no player has an incentive to unilaterally change his/her action. Now, we consider the following assumption.

Assumption (H): Given the distribution of the integer random variable K, the payoff function

$$x \longmapsto D(x) = [f(1,x), f(2,x), \ldots, f(N,x)]$$

is continuous on M.

Note that (H) is satisfied for any integer random variable with finite support. Let $proj_M$ be the projection into the set M defined as $proj_M(y) = \arg\min_{x \in M} \ \|y - x\|_2$, where $\|y\|_2 = \sqrt{y_1^2 + \cdots + y_N^2}$ is the Euclidean norm of y. Since M is a nonempty, convex, and compact subset of $\mathbb{R}^N$, the projection $proj_M(y)$ is unique (there is a unique minimizer on M of the function $x \longrightarrow \|y - x\|_2$).

Theorem 1: Under assumption (H), the evolutionary game with a random number of interacting users has at least one equilibrium.

Proof: We show that there exists a probability vector $x \in M$ such that the inequality $\sum_{j \in \mathcal{A}} (x_j - mut_j) f(j,x) \geq 0$ holds for all $mut \in M$. Let $\theta > 0$. The problem is equivalent to the existence of a solution of the variational inequality problem: find $x \in M$ such that $\forall \, mut$

$$\langle x - mut, \theta D(x) \rangle = \sum_{j \in \mathcal{A}} (x_j - mut_j) \theta f(j,x) \geq 0$$

where θ is a positive value. The term $\langle x - mut, \theta D(x) \rangle$ can be rewritten as $\langle x - mut, (x + \theta D(x)) - x \rangle$. Thus, the equilibrium is a solution of

$$\langle x - mut, (x + \theta D(x)) - x \rangle \leq 0, \qquad \forall \, mut.$$

This implies that $x = proj_M(x + \theta D(x))$. It is known that $proj_M$ is a 1-Lipschitz function (hence, continuous). Since D is a continuous function, the composition $proj_M(Id + \theta D)$ is continuous, and M is a nonempty, convex, and compact subset of $\mathbb{R}^N$. Using the Brouwer fixed-point theorem, the map $proj_M(Id + \theta D)$ has at least one fixed point x^* in M. x^* is our desired equilibrium. ■

This result can be extended into the sequential diagonal transfer continuous function case.

Evolutionary Game Dynamics: Evolutionary game theory considers a dynamic scenario where players are interacting with others players and adapting their choices based on the fitness they receive. A strategy having higher fitness than others tends to gain ground: This is formulated through rules describing the dynamics (such as the replicator dynamics or others) of the sizes of populations (of strategies).

Replicator Dynamics: Replicator dynamics is one of the most studied dynamics in evolutionary game theory. It has been introduced by Taylor and Jonker [6]. The replicator dynamics has been used for describing the evolution of road traffic congestion in which fitness is determined by the strategies chosen by all drivers [4]. It has also been studied in the context of the association problem in wireless communications [11] and evolutionary network formation and fuzzy coalition in heterogeneous networks [12]. We introduce the replicator dynamics that describes the evolution in the population of various strategies. In the replicator dynamics, the share of a strategy j in the population grows at a rate that is proportional to the difference between the payoff of that strategy and the average payoff of the population. The replicator dynamic equation is given by

$$\dot{x}_j(t) = \mu \, x_j(t) \left[f\left(j, x(t)\right) - \sum_{l=1}^{N} x_l(t) f\left(l, x(t)\right) \right] \quad (2)$$

where μ is some positive constant. The parameter μ can be used to tune the rate of convergence, and it may be interpreted as the rate that a player of the population participates in a (local interaction) game. In biology, it can represent the probability that an animal finds a resource available.

It is known from [3] and [4] that Lyapunov stability under a replicator also shows equilibrium behavior, and the solutions of the replicator have been shown to converge to the set of Nash equilibria in important classes of games (e.g., potential games, zero-sum games, supermodular games, and some classes of stable games). However, in general, the solutions of replicator dynamics need not converge to the set of equilibria [3].

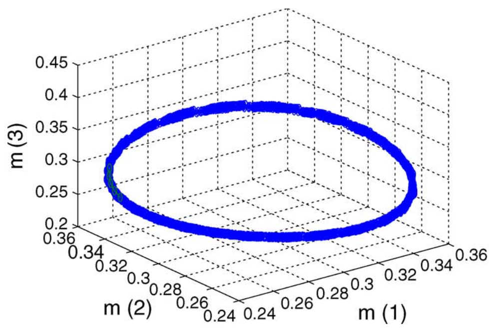

Fig. 4. Case a. (Left) $(\mu_1, \mu_2) = (1, 1)$. (Right) $(\mu_1, \mu_2) = (1.9, 2)$.

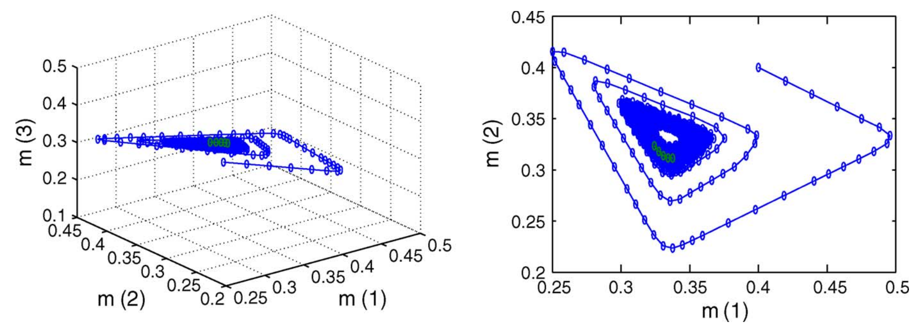

Fig. 5. Case b. Same parameters as in case a. (Right) $\mu_3 = 45$; existence of a cycle limit.

Other Evolutionary Game Dynamics: There are a large number of population dynamics other than the replicator dynamics that have been used in the context of noncooperative games. Examples are excess payoff dynamics, fictitious play dynamics, gradient methods, generalized Smith dynamics, generating function-based dynamics, projection dynamics, imitate-the-better dynamics, spatial mean dynamics with diffusion, and evolutionary game dynamics with migration. Much literature can be found in the extensive survey on evolutionary game dynamics in [3] and [4].

Nonconvergent Behaviors Under Mean Dynamics: It is known that a specific payoff structure such as the structure of potential, stable, and supermodular games makes evolutionary justifications of the equilibrium prediction. However, once we move beyond these particular classes of population games, it is not clear how often convergence will occur. In addition, most of the games do not have these specific properties. The following examples counterbalance the convergence approach by investigating nonconvergence of evolutionary dynamics for games, describing situations in which cycling offers the best predictions of long run behavior.

We illustrate nonconvergent behaviors with the following payoff matrix:

$$\left(\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & \mu_1 & -\mu_2 \\ 2 & -\mu_2 & 0 & \mu_1 \\ 3 & \mu_1 & -\mu_2 & 0 \end{array}\right)$$

under the mean dynamics

$$\dot{m}_i = \sum_j m_j L_{ji}(m) - m_i \sum_j L_{ij}(m).$$

We examine two cases: 1) replicator dynamics, i.e., $L_{ij}(m) = m_j \max(0, f(j,m) - f(i,m))$, and 2) Boltzmann–Gibbs (logit) dynamics, i.e., $L_{ij} = (e^{\mu_3 f(j,m)})/(\sum_{j' \in \{1,2,3\}} e^{\mu_3 f(j',m)})$ (Figs. 4 and 5).

Case (a): Trajectories for $(\mu_1, \mu_2) \in \{(1,1), (1.9, 2)\}$.

Case (b): Trajectories for $\mu_3 = 45$.

Delayed Evolutionary Game Dynamics: The fitness for a player at a given time is determined by action i taken by the player at that time, as well as by the actions of the population it interacts with, that was taken τ_i units ago. More precisely, if an anonymous user 1 chooses the strategy j at time t when the population profile is x, then user 1 will receive the payoff $f(j, x(t))$ only τ_k times later. Thus, the payoff at time t is given by $f(j, x(t - \tau_j))$. In the replicator dynamics with time delays, the share of a strategy j in the population grows at a rate that is proportional to the difference between the payoff of that strategy delayed by an average time delay τ_j and the average delayed payoff of the population. The replicator dynamic equation is then given by

$$\dot{x}_j(t) = \mu\, x_j(t) \left[f\left(j, x(t - \tau_j)\right) - \sum_{l=1}^{N} x_l f\left(l, x(t - \tau_l)\right) \right].$$

The parameters τ_j and μ do not change the evolutionary stable strategies set but have a big influence on the stability of the system.

IV. Slotted ALOHA-Based Access Network

Here, we consider an Aloha system in which mobiles make transmission decisions in an effort to maximize their utility. We assume that mobiles are randomly placed over a plane. All mobiles use the same fixed transmission range of r. The channel is ideal for transmission, and all errors are due to collision. A mobile decides to transmit a packet or not to transmit to a receiver when they are within a transmission range of each other. Interference occurs as in the Aloha protocol: If more than one neighbor of a receiver transmits a packet at the same time, then there is a collision. Let μ be the probability that a mobile i has its receiver $R(i)$ within its range. When a mobile i transmits to $R(i)$, all mobiles within a circle of radius R centered at $R(i)$ cause interference to the node i for its transmission to $R(i)$. This means that more than one transmission within a distance R of the receiver in the same slot causes a collision and the loss of mobile i's packet at $R(i)$.

Each mobile has two possible strategies: either to transmit (T) or to stay quiet (S). If mobile i transmits a packet, it incurs a transmission cost of $\delta \geq 0$. The packet transmission is successful if the other users do not transmit (stays quiet) in that given time slot; otherwise, there is a collision, and the corresponding cost is $\Delta \geq 0$. If there is no collision, user i gets a reward of V from the successful packet transmission. We suppose that the payoff V is greater than the cost of transmission δ. When all users stay quiet, they have to pay a regret cost κ. If $\kappa = 0$, the game is called *degenerate multiple-access game*. The ESS corresponding to any number of nodes[1] of this game is given in Theorem 2.

A. Utility Function and ESS

Let $\mathcal{A} := \{T, S\}$ be the set of strategies. An equivalent interpretation of strategies is obtained by assuming that individuals choose pure strategies, and then the probability distribution represents the fraction of individuals in the population that choose each strategy. We denote by s (resp. $1-s$) the population share of strategy T (resp. S). The payoff obtained by a node with k other interfering nodes when it plays T is $u_k(T, s) = (-\Delta - \delta)(1 - \eta_k) + (V - \delta)\eta_k$, where $\eta_k := (1-s)^k$. The node mutant receives $u_k(S, s) = -\kappa(1-s)^k$ when it stays quiet. The expected payoff of an anonymous transmitter node mutant is given by

$$\begin{aligned} f(T,s) &= \mu \sum_{k \geq 0} \mathbb{P}(K = k) u_k(T, s) \\ &= -\mu(\Delta + \delta) + \mu(V + \Delta) G_K(1 - s) \end{aligned}$$

where G_K is the generating function of K. Analogously, we have

$$f(S, s) = -\mu\kappa \sum_{k \geq 0} (1-s)^k \mathbb{P}(K = k).$$

B. Existence and Uniqueness of the ESS

We introduce two alternative information scenarios that have an impact on decision making. Thus, we will study three different scenarios as follows.

[1]The one-shot game with n nodes has $2^n - 1$ Nash equilibria and a unique ESS.

Case 1) The mobile does not know whether there are zero or more other mobiles in a given local interaction game about to be played.

Case 2) The mobile knows if there is a transmitter at the range of its receiver. Consequently, he transmits with probability 1 in case no other potential interferers are present.

Case 3) "Massively dense." The mobile is never alone in transmitting during a slot. That means there is at least one other mobile that is involved in the local interaction game.

We denote $\alpha := (\Delta + \delta)/(V + \Delta + \kappa)$, which represents the ratio between the collision cost $-\Delta - \delta$ (the cost when there is a collision during a transmission) and the difference between the global cost perceived by a mobile $-\Delta - \delta - \kappa$ (collision and regret) and the benefit $V - \delta$ (reward minus transmission cost). When the collision cost Δ becomes high, the value α converges to one, and when the reward or regret cost becomes high, the value α is close to zero.

A transmitter does not know if there are other transmitters at the range of its receiver. Then, even when it is the only transmitter, it has to decide whether to transmit or not.

Theorem 2:

Case 1) If $\mathbb{P}(K = 0) < \alpha$, then the game has a unique ESS s_1^* given by $s_1^* = \phi^{-1}(\alpha)$, where $\phi : s \mapsto \sum_{k \geq 0} \mathbb{P}(K = k)(1 - s)^k$.

Case 2) An anonymous user without an interfering user receives the fitness $V - \delta$. If $\mathbb{P}(K = 0) < ((\Delta + \delta)/(V + \Delta))$, then the game has a unique ESS s_2^* given by $s_2^* = \phi^{-1}((\Delta + \delta + \kappa\mathbb{P}(K = 0))/(V + \Delta + \kappa))$, where $\phi : s \mapsto \sum_{k \geq 0} \mathbb{P}(K = k)(1 - s)^k$.

Case 3) The game always has a unique ESS that is a solution of the following equation: $\sum_{k \geq 1} \mathbb{P}(K = k)(1 - s)^k = \alpha$.

Proof: A strictly mixed equilibrium s is characterized by $f(T, s) = f(S, s)$, i.e., $\phi(s) = \alpha$. The function ϕ is continuous and strictly decreasing monotone on (0, 1), with $\phi(1) = \mathbb{P}(K = 0)$ and $\phi(0) = 1$. Then, the equation $\phi(s) = ((\Delta + \delta)/(V + \Delta + \kappa))$ has a unique solution in the interval $(\mathbb{P}(K = 0), 1)$. Thus, we have

$$f(s, y) - f(mut, y) = \mu(V + \Delta + \kappa)(s - mut)\left(\phi(y) - \phi(s)\right).$$

Since $s - \epsilon mut - (1-\epsilon)s = \epsilon(s - mut)$, for $y = \epsilon mut + (1-\epsilon)s$, one has

$$\sum_{j \in \{T,S\}} (x_j - mut_j) f(j, y) > 0$$

(because ϕ is a strictly decreasing continuous function) for all $mut \neq s$. This completes the proof. ■

C. Spatial Node Distribution and the ESS

In this part, we study two cases of a spatial node distribution. In the first one, we assume that the number of nodes in a local interaction is fixed, and, in the second one, we assume that nodes are distributed over a plane following a Poisson distribution. These allow us to explicitly compute the ESS and propose some optimization issues in slotted Aloha-based wireless networks.

1) Fixed Number of Nodes in a Local Interaction: In this part, we suppose that the population of nodes is composed of many local interactions between n nodes, where $n \geq 2$. Let $\mathcal{A} := \{T, S\}$ be the set of strategies, and assume that the strategy T has a delay τ_T and the strategy S has a delay τ_S. The payoff of a player using the action $a_i \in \mathcal{A}$ against the other players when they use the multistrategy $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ is given by $U_i(a)$. Each user plays the n-player following game $\Gamma_n = (\mathcal{N}, \mathcal{A}, (U_i)_{i \in \mathcal{N}})$, where 1) $\mathcal{N}$ is the set of users (nodes), in which the cardinal of $\mathcal{N}$ is n; 2) $\mathcal{A}$ is the set of pure actions (the same for every user); and 3) for every user i in $\mathcal{N}$, the payoff function $U_i : \mathcal{A}^n \rightarrow \mathbb{R}$ is given by

$$U_i(a) = \begin{cases} V - \delta, & \text{if } a_i = T \text{ and } a_j = S, \quad \forall\, j \neq i \\ 0, & \text{if } a_i = S \text{ and } \{j \in \mathcal{N} \mid a_j = T\} \geq 1 \\ -\Delta - \delta, & \text{if } a_i = T \text{ and } \{j \in \mathcal{N} \mid a_j = T\} \geq 2 \\ -\kappa, & \text{if } a_j = S, \quad \forall\, j \in \mathcal{N}. \end{cases}$$

Let s be the proportion of nodes in the population using the strategy T. Then, $x = (s, 1-s)$ is the state of the population. Let $\Delta(\mathcal{A}) := \{sT + (1-s)S | 0 \leq s \leq 1\}$ be the set of mixed strategies. The average payoff is

$$f(s,s) = \mu s \left[(-\Delta - \delta)\left(1 - (1-s)^{n-1}\right) + (V - \delta)(1-s)^{n-1}\right] - \mu\kappa(1-s)^n.$$

It is not difficult to see that the one-shot game Γ_n has $2^n - 1$ Nash equilibrium; n of them are optimal in the Pareto sense.[2]

- If only one node *transmits* and the others *stay quiet*, then the node that transmits gets the payoff $V - \delta$, and the others receive nothing and have no cost. This configuration is an equilibrium.
- There are exactly n pure equilibria, and all these pure equilibria are Pareto optimal.
- k $(1 \leq k < n-1)$ of the n nodes choose to stay quiet, and the $n-k$ others are active and play the optimal mixed strategy in the game Γ_{n-k}: $(1 - \alpha^{1/(n-k-1)}, \alpha^{1/(n-k-1)})$, where $\alpha := ((\Delta + \delta)/(V + \Delta + \kappa))$. Thus, there are exactly $\sum_{k=1}^{n-2} \binom{n}{k} = 2^n - (n+2)$ partially mixed Nash equilibria.
- The game has a unique strictly mixed Nash equilibrium given by $(1 - \alpha^{1/(n-1)}, \alpha^{1/(n-1)})$.
- The allocation of payoff obtained in this (partially or completely) mixed strategy is not Pareto optimal.

Note that the first interference scenario described in the previous section holds here because the number of interferers is fixed and is equal to $n-1$. Then, from Theorem 2 with the function $\phi(s) = (1-s)^{n-1}$, an ESS exists and is uniquely defined by $s^* = 1 - \alpha^{1/(n-1)}$. This result generalizes the ESS in the two-player case that we have shown in [9].

[2] An allocation of payoffs is Pareto optimal or Pareto efficient if there is no other allocation that makes every node at least as well off and at least one node strictly better off.

2) Poisson Distribution: We consider that nodes are distributed over a plan following a Poisson distribution with density λ. The probability that a node has k neighbors is given by the following distribution.

Cases 1 and 2: $\mathbb{P}(K = k) = ((\lambda\pi r^2)^k / k!) e^{-\lambda\pi r^2}$, $k \geq 0$.
Case 3: $\mathbb{P}(K = k) = ((\lambda\pi r^2)^{k-1} / (k-1)!) e^{-\lambda\pi r^2}$, $k \geq 1$.

Considering those node distributions and from previous theorems, the unique ESS s^* for all cases is the solution of the following equation:

$$\begin{cases} e^{-\lambda\pi r^2 s_1} = \alpha, & \text{for case 1} \\ e^{-\lambda\pi r^2 s_2} = \alpha + \frac{\kappa\mathbb{P}(K=0)}{V+\Delta+\kappa}, & \text{for case 2} \\ (1 - s_3) e^{-\lambda\pi r^2 s_3} = \alpha, & \text{for case 3.} \end{cases}$$

Thus, we obtain the following equilibria in the different scenario:

$$s_1^* = \log\left(\alpha^{-\frac{1}{\lambda\pi r^2}}\right)$$

$$s_2^* = \log\left(\left(\alpha + \frac{\kappa\mathbb{P}(K=0)}{V+\Delta+\kappa}\right)^{-\frac{1}{\lambda\pi r^2}}\right)$$

$$\text{and } s_3^* = 1 - \frac{\mathrm{LambertW}(\lambda\pi r^2 \alpha e^{\lambda\pi r^2})}{\lambda\pi r^2}$$

$$(\mathrm{LambertW}(s) \text{ is the inverse of } f(w) = we^w).$$

D. Optimization Issue

Here, we discuss some optimization issues that can be attained by changing the cost parameters. We look for the probability of success that can be achieved in a local interaction, depending on the node distribution and also cost parameters.

1) Fixed Number of Nodes in a Local Interaction: We assume here that every mobile has the same number of interfering users, that is, $n-1$. At the equilibrium point, the probability of success $P_{\mathrm{succ}}(n)$ of a node is given by $s^*(1-s^*)^{n-1}$. The total probability to have a successful transmission in a local interaction, which we denote later as β, is given by

$$\beta(\alpha, n) = n\mu s^*(1-s^*)^{n-1} = n\mu\left(1 - \alpha^{\frac{1}{n-1}}\right)\alpha \qquad (3)$$

where μ is the probability that a mobile has a receiver in its range. The total throughput $\beta(\alpha, n)$ goes to $-\mu\alpha\log(\alpha)$ when the number of nodes n goes to infinity. Hence, when n is very large, the total throughput is maximized when the cost ratio is $\alpha = 1/e$. Then, the total throughput $\beta(1/e, n)$ tends to the value μ/e when the number of nodes tends to infinity.

For a fixed number of nodes n, the optimal total throughput is obtained when $\alpha^* = (1 - 1/n)^{n-1}$, and the corresponding total throughput converges to the value μ/e when the number of nodes tends to infinity, i.e.,

$$\lim_{n\to\infty} \beta(\alpha^*, n) = \lim_{n\to\infty} \mu\left(1 - \frac{1}{n}\right)^{n-1} = \frac{\mu}{e}.$$

The optimal total throughput with an infinite number of nodes is μ/e, which is the product of the probability μ for a node to have a receiver in its range and the maximum throughput $1/e$ of a slotted Aloha system with an infinite number of nodes.

2) Poisson Distribution: We look for the average total throughput that can be achieved in a local interaction depending on distribution parameters and also cost parameters. We consider the Poisson distribution with parameters λ and r. The average total throughput is given by

$$\beta(\alpha,\lambda) = \mu \sum_{k\geq 0} k\mathbb{P}(K=k) P_{\text{succ}}(k)$$

where $P_{\text{succ}}(k)$ is the probability of success for one node when the number of nodes is k, which depends on the scenario considered. Then, the total throughput that can be achieved in a local interaction is given by a different equation depending on the scenario considered. In case 1, we have

$$\begin{aligned}\beta(\alpha,\lambda) &= \mu s_1^* \sum_{k\geq 0} k\mathbb{P}(K=k)\left(1-s_1^*\right)^k \\ &\approx \mu s_1^*\left(1-s_1^*\right)\lambda\pi r^2\alpha.\end{aligned}$$

In case 2, we have

$$\beta(\bar{\alpha},\lambda) \approx \mu s_2^*\left(1-s_2^*\right)\lambda\pi r^2\left(\alpha + \frac{\kappa\mathbb{P}(K=0)}{V+\Delta+\kappa}\right).$$

We derive immediately the following result.

Proposition 1: The maximum total throughput under a Poisson distribution is attained when $\alpha = e^{h(\lambda,r)}$ in case 1 [resp. $\alpha = e^{h(\lambda,r)} - ((\kappa\mathbb{P}(K=0))/(V+\Delta+\kappa))$ in case 2], where h is one of the two functions defined by

$$(\lambda,r)\in\mathbb{R}_+^2 \mapsto \frac{-(1+2\lambda\pi r^2)\pm\sqrt{1+4(\lambda\pi r^2)^2}}{2}.$$

In case 3, we have $\beta(\alpha,\lambda) = \mu s_3^* \sum_{k\geq 1} kP(K=k)(1-s_3^*)^k \approx \mu\alpha s_3^*(1+\lambda\pi r^2(1-s_3^*))$. In the following proposition, we give the optimal throughput in case 3.

Proposition 2: In case 3, there exists a unique α_3^* in which the total throughput is maximized. α_3^* is given by $\alpha_3^* = (1-s)e^{-\lambda\pi r^2 s}$, where s is the unique solution in [0, 1] of the following:

$$1+\gamma - s(2+5\gamma+\gamma^2) + s^2(4\gamma+2\gamma^2) - \gamma^2 s^3 = 0.$$

Proof: The derivative of the function $H := (\partial\beta/\partial s)$ is given by

$$H(s) = \left(1+\gamma - s(2+5\gamma+\gamma^2)+s^2(4\gamma+2\gamma^2) - s^3\gamma^2\right)e^{-\gamma s}.$$

We prove that the above function is strictly decreasing in [0, 1]. For that, it is sufficient to study the following function:

$$G(s) = 1+\gamma - s(2+5\gamma+\gamma^2) + s^2(4\gamma+2\gamma^2) - s^3\gamma^2.$$

We have $\partial G(s)/\partial s$, which is given by

$$\frac{\partial G(s)}{\partial s} = -(2+5\gamma+\gamma^2) + 2s(4\gamma+2\gamma^2) - 3s^2\gamma^2.$$

It is easy to show that the above function is always negative. Since $H(0) = 1+\gamma > 0$ and $H(1) = -e^{-\gamma} < 0$, then the function H is positive for $s\in[0,\bar{s})$ and is negative for $s\in(\bar{s},1]$, where $\bar{s}$ is the solution of the equation $G(s)=0$. Since s^* is a decreasing function of α, we conclude that the function P_{succ} is positive if $s\in[0,\bar{s})$ and is negative if $s\in(\bar{s},1]$, since the optimum of function P_{succ} is attained at $\alpha = (1-\bar{s})e^{-\lambda\pi r^2\bar{s}}$. ■

E. Evolutionary Game Dynamics

Evolutionary game dynamics gives a tool for observing the evolution of strategies in the population in time. The most famous one is the replicator, based on replication by imitation, which we consider here for observing the evolution of the strategies T and S in the population of nodes.

Proposition 3: The ESS given in Theorem 2 is asymptotically stable in the replicator dynamics without delays for all nontrivial initial state ($s_0 \notin \{0,1\}$).

Proof: The replicator dynamics is given by

$$\frac{d}{dt}s(t) = \mu(V+\Delta+\kappa)s(t)\left(1-s(t)\right)\left(\phi\left(s(t)\right)-\alpha\right).$$

The function ϕ is decreasing in (0, 1), which implies that the derivative of the function $s(1-s)(\phi(s)-\alpha)$ at the ESS is negative. Hence, the ESS is asymptotically stable. ■

Now, we study the effect of the time delays on the convergence of replicator dynamics to the ESSs in which each pure strategy is associated with its own delay. Let τ_T (resp. τ_S) be the time delay of the strategy (T) [resp. (S)]. The replicator dynamic equation given in (2) becomes

$$\dot{s}(t) = \mu\, s(t)\left(1-s(t)\right)\left[f\left(T,s(t-\tau_T)\right) - f\left(S,s(t-\tau_S)\right)\right] \tag{4}$$

where

$$\begin{aligned} f\left(T,s(t)\right) &:= -\mu(\Delta+\delta)\left(1-\left(1-s(t)\right)^{n-1}\right) \\ &\quad + \mu(V-\delta)\left(1-s(t)\right)^{n-1}, \\ f\left(S,s(t)\right) &:= -\mu\kappa\left(1-s(t)\right)^{n-1}. \end{aligned}$$

To study the asymptotic stability of the replicator dynamics (4) around the unique ESS $s_1^* = 1-(\Delta+\delta/V+\Delta+\kappa)^{1/(n-1)}$, we linearize (4) at s_1^*. We obtain the following linear delay differential equation:

$$\begin{aligned}\dot{z}(t) = &-\mu(n-1)s_1^*\left(1-s_1^*\right)^{n-1} \\ &\times\left((V+\Delta)z(t-\tau_T)+\kappa z(t-\tau_S)\right)\end{aligned} \tag{5}$$

where $z(t) = s(t)-s_1^*$. The following theorem gives sufficient conditions of stability of (5) at zero.

Theorem 3: Suppose that at least one of the following conditions holds.

- $(V+\Delta)\tau_T + \kappa\tau_S < (1/(n-1)s(1-s)^{n-1}\mu)$;
- $V+\Delta > \kappa$ and $(V+\Delta)\tau_T < ((V+\Delta-\kappa)/((n-1)s(1-s)^{n-1}\mu(V+\Delta+\kappa)))$;
- $V+\Delta < \kappa$ and $\kappa\tau_S < ((-V-\Delta+\kappa)/((n-1)s(1-s)^{n-1}\mu(V+\Delta+\kappa)))$.

Then, the ESS s is asymptotically stable.

A proof of Theorem 3 can be obtained using , th. 3[8] and applying it to (5).

We are looking for a necessary and sufficient condition of stability of the differential equation (5). For finding this, we need the following lemma.

Lemma 1 ([7]): The trivial solution of the linear delay differential equation $\dot{z}(t) = -az(t-\tau)$, $\tau, a > 0$, is asymptotically stable if and only if $2a\tau < \pi$.

Given this lemma, a necessary and sufficient condition of stability of (5) at zero when delays are symmetric is given in the following theorem.

Theorem 4 (Symmetric Delay): Suppose that $\tau_T = \tau_S = \tau$. Then, the ESS s_1^* is asymptotically stable if and only if $\tau < (\pi/2(n-1)\mu s_1^*(1-s_1^*)^{n-1}(V+\Delta+\kappa))$.

Proof: By applying symmetric delay $\tau_T = \tau_S = \tau$ in (5), one has

$$\dot{z}(t) = -\mu(n-1)s_1^*\left(1-s_1^*\right)^{n-1}\left(V+\Delta+\kappa\right)z(t-\tau). \quad (6)$$

We then apply Lemma 1 for the parameter $a = \mu(n-1)s_1^*(1-s_1^*)^{n-1}(V+\Delta+\kappa) > 0$. ■

V. W-CDMA Wireless Networks

Here, we apply evolutionary games to noncooperative power control in wireless networks. Specifically, we focus our study in uplink power control in W-CDMA wireless systems. Here, the random number of interfering mobiles with a given randomly selected mobile is induced by the geographical position of the mobiles compared with the base stations.

A. Decentralized Power Control

Here, we study competitive decentralized power control in a wireless network where the mobiles use, as an uplink medium-access-control protocol, the W-CDMA technique to transmit to a base station. We assume that there is a large population of mobiles that are randomly placed over a plane following a Poisson process with density λ. We consider a random number of mobiles interacting locally. When a mobile i transmits to its receiver $R(i)$, all mobiles cause interference to the transmission from node i to receiver $R(i)$. We assume that a mobile is within a circle of a receiver with probability μ. We define a random variable $\mathcal{R}$ that will be used to represent the distance between a mobile and a receiver. Let $\varsigma(r)$ be the probability density function for $\mathcal{R}$. Then, we have $\mu = \int_0^R \varsigma(r)dr$.

Remark 1: If we assume that the receivers or access points are randomly distributed following a Poisson process with density ν, the probability density function is expressed by $\varsigma(r) = \nu e^{-\nu r}$.

For uplink transmissions, a mobile has to choose between high (H) power level named P_H and low (L) power level named P_L. Let s be the population share strategy H. Hence, the signal P_r received at the receiver from a mobile is given by $P_r = gP_i l(r)$, where g is the gain antenna, $P_i \in \{P_L, P_H\}$ is the power level used by the mobile, and r is the distance from the mobile to the base station. For the attenuation, the most common function is $l(t) = 1/t^{\alpha}$, with α ranging from 3 to 6. Note that such $l(t)$ explodes at $t = 0$ and, thus, in particular, is not correct for a small distance r and large intensity λ. Then, it makes sense to assume attenuation to be a bounded function in the vicinity of the antenna. Hence, the last function becomes $l(t) = \max(t, r_0)^{-\alpha}$. First, we note that the number of transmissions within a circle of radius r_0 centered at the receiver is $\lambda\pi r_0^2$. Then, the interference caused by all mobiles in that circle is $I_0(s) = (\lambda\pi g(sP_H + (1-s)P_L)/r_0^{\alpha-2})$.

Now, we consider a thin ring $\mathcal{A}_j$ with the inner radius $r_j = jdr$ and the outer radius $r_j = r_0 + jdr$. The signal power received at the receiver from any node in $\mathcal{A}_j$ is $P_{r_i} = gP_i/r_i^{\alpha}$. Hence, the interference caused by all mobiles in $\mathcal{A}_j$ is given by

$$I_j(s) = \begin{cases} 2g\lambda\pi r_j dr\left(\frac{sP_H+(1-s)P_L}{r_j^{\alpha}}\right), & \text{if } r_j < R, \\ 2\mu g\lambda\pi r_j dr\left(\frac{sP_H+(1-s)P_L}{r_j^{\alpha}}\right), & \text{if } r_j \geq R. \end{cases}$$

Hence, the total interference contributed by all nodes at the receiver is

$$\begin{aligned} I(s) &= I_0(s) + 2g\lambda\pi\left(sP_H + (1-s)P_L\right) \\ &\quad \times \left[\int_{r_0}^{R} \frac{1}{r^{\alpha-1}}\,dr + \mu\int_{R}^{\infty}\frac{1}{r^{\alpha-1}}\,dr\right] \\ &= g\lambda\pi\left(sP_H + (1-s)P_L\right) \\ &\quad \times\left(\frac{\alpha}{\alpha-2}r_0^{-(\alpha-2)} - 2(1-\mu)R^{-(\alpha-2)}\right). \end{aligned}$$

Hence, the signal-to-interference-plus-noise ratio (SINR) is given by

$$\mathrm{SINR}_i(P_i, s, r) = \begin{cases} \frac{gP_i/r_0^{\alpha}}{\sigma+\beta I(s)} & \text{if } r \leq r_0, \\ \frac{gP_i/r^{\alpha}}{\sigma+\beta I(s)} & \text{if } r \geq r_0 \end{cases}$$

where σ is the power of the thermal background noise, and β is the inverse of the processing gain of the system. This parameter weighs the effect of interference depending on the orthogonality between codes used during simultaneous transmissions. In the sequel, we compute the mobile's utility (fitness) depending not only on his decision but also on the decision of his interferers. We assume that the user's utility (fitness) choosing power level P_i is expressed by

$$f(P_i, s) = w\int_0^R \log\left(1 + SINR(P_i, s, r)\right)\varsigma(r)dr - \eta P_i.$$

The pricing function P_i defines the instantaneous "price" that a mobile pays for using a specific amount of power that causes interference in the system; w and η are cost parameters. The parameter η can be the power cost consumption for sending packets.

We are now looking at the existence and uniqueness of the ESS. For this, we need the following result.

Lemma 2: For all density function ς defined in $[0, R]$, the function $h : [0,1] \to \mathbb{R}$ defined as $s \longmapsto \int_0^R \log(1 + \mathrm{SINR}(P_H, s, r)/1 + \mathrm{SINR}(P_L, s, r))\varsigma(r)\,dr$ is continuous and strictly monotonic.

Proof: The function

$$s \longmapsto \log\left(\frac{1+\mathrm{SINR}(P_H,s,r)}{1+\mathrm{SINR}(P_L,s,r)}\right)\varsigma(r)$$

is continuous and integrable in r in the interval $[0,R]$. The function h is continuous. Using derivative properties of an integral with a parameter, we can see that the derivative function of h is the function $h' : [0,1] \to \mathbb{R}$ defined as

$$s \longmapsto \int_0^R \frac{\partial}{\partial s}\left[\log\left(\frac{1+\mathrm{SINR}(P_H,s,r)}{1+\mathrm{SINR}(P_L,s,r)}\right)\right]\varsigma(r).$$

We show that the term

$$\frac{\partial}{\partial s}\left[\log\left(\frac{1+\mathrm{SINR}(P_H,s,r)}{1+\mathrm{SINR}(P_L,s,r)}\right)\right]$$

is negative. Let $W(s) := (1+\mathrm{SINR}(P_H,s,r))/(1+\mathrm{SINR}(P_L,s,r))$. The function W can be rewritten as $W(s) = 1 + ((g(P_H-P_L)/r^\alpha)/(\sigma+\beta I(s)+gP_L/r^\alpha))$, where $I(s) = (s(P_H-P_L)+P_L)c(r)$ and $c(r) = \lambda\pi g[(\alpha/\alpha-2)r_0^{-(\alpha-2)} - 2(1-\mu)R^{-(\alpha-2)}]$ if $r \geq r_0$ and $\lambda\pi g/r_0^{\alpha-2}$ otherwise. Since W satisfies $W(s) > 1$ and $W'(s) = -c(r)\beta(P_H-P_L)((g(P_H-P_L)/r^\alpha)/((\sigma+\beta I(s)+gP_L/r^\alpha)^2)) < 0$, hence

$$\frac{\partial}{\partial s}\left[\log\left(\frac{1+SINR(P_H,s,r)}{1+SINR(P_L,s,r)}\right)\right] = \frac{W'(s)}{W(s)} < 0$$

i.e., $h'(s) < 0$. Hence, h is strictly decreasing. ∎

Proposition 4: For all density function ς, the pure strategy P_H dominates the strategy P_L if and only if $(\eta/w)(P_H-P_L) < \int_0^R \log((1+\mathrm{SINR}(P_H,P_H,r))/(1+\mathrm{SINR}(P_L,P_H,r)))\varsigma(r)\,dr = h(1)$. For all density function ς, the pure strategy P_L dominates the strategy P_H if and only if $(\eta/w)(P_H-P_L) > \int_0^R \log((1+\mathrm{SINR}(P_H,P_L,r))/(1+\mathrm{SINR}(P_L,P_L,r)))\varsigma(r)\,dr = h(0)$.

Proof: We decompose the existence of the ESS in several cases. 1) P_H is preferred over P_L: The higher power level dominates the lower if and only if $f(P_H,P_H) > f(P_L,P_H)$ and $f(P_H,P_L) > f(P_L,P_L)$. These two inequalities imply

$$\frac{\eta}{w}(P_H-P_L) < \int_0^R \log\left(\frac{1+\mathrm{SINR}(P_H,P_H,r)}{1+\mathrm{SINR}(P_L,P_H,r)}\right)\varsigma(r)dr.$$

2) P_L is preferred over P_H: Analogously, the lower power dominates the higher power if and only if $f(P_L,P_H) > f(P_H,P_H)$ and $f(P_L,P_L) > f(P_H,P_L)$, i.e., $(\eta/w)(P_H-P_L) > \int_0^R \log((1+\mathrm{SINR}(P_H,P_L,r))/(1+\mathrm{SINR}(P_L,P_L,r)))\varsigma(r)\,dr$. ∎

The following result gives sufficient conditions for existence and uniqueness of the ESS in the W-CDMA uplink power control.

Proposition 5: For all density function ς, if $h(1) < (\eta/w)(P_H-P_L) < h(0)$, then there exists a unique ESS s^* that is given by $s^* = h^{-1}((\eta/w)(P_H-P_L))$.

Proof: Suppose that the parameters w, η, P_H, and P_L satisfy the following inequality: $h(1) < (\eta/w)(P_H-P_L) < h(0)$. Then, the game has no dominant strategy. A mixed equilibrium is characterized by $f(P_H,s) = f(P_L,s)$. It is easy to see that this last equation is equivalent to $h(s) = (\eta/w)(P_H-P_L)$. From Lemma 2, we have that the equation $h(s) = (\eta/w)(P_H-P_L)$ has a unique solution given by $s^* = h^{-1}((\eta/w)(P_H-P_L))$. We now prove that this mixed equilibrium is an ESS. To prove this result, we compare $s^*f(P_H,mut) + (1-s^*)f(P_L,mut)$ and $mutf(P_H,mut) + (1-mut)f(P_L,mut)$ for all $mut \neq s^*$. The difference between the two values is exactly $w(s^*-mut)(h(mut)-h(s^*))$. According to Lemma 2, h is a decreasing function. Hence, $(s^*-mut)(h(mut)-h(s^*))$ is strictly positive for all strategy mut different from s^*. We conclude that the mixed equilibrium $(s^*, 1-s^*)$ is an ESS. ∎

From the last proposition, we can use the pricing parameter η as a design tool for creating an incentive for the user to adjust their power control. We observe that the ESS s^* decreases when η increases. This means that the mobiles become less aggressive as the pricing function increases, and the system can limit aggressive requests for the SINR.

B. Evolutionary Game Dynamics in W-CDMA

Next, we use the replicator dynamics for observing the evolution of the strategies P_H and P_L in the population of nodes. We study the effect of the time delays on the convergence of the replicator dynamics to the ESSs in which each pure strategy is associated with its own delay. Let τ_H (resp. τ_L) be the time delay of the strategy P_H (resp. P_L). The delayed replicator dynamic equation given in (2) becomes

$$\dot{s}(t) = \mu s(t)\,(1-s(t))\,\widehat{\Delta}(t) \tag{7}$$

where $\widehat{\Delta}(t) := f(P_H, s(t-\tau_H)) - f(P_L, s(t-\tau_L))$.

Proposition 6: The ESS $s^* = h^{-1}((\eta/w)(P_H-P_L))$ is asymptotically stable under the replicator dynamics without time delays for all nontrivial initial state.

Proof: The replicator dynamics without time delays is given by $\dot{s}(t) = \mu w s(t)(1-s(t))(h(s(t)) - \eta(P_H-P_L)/w)$. The function h is decreasing in (0, 1), which implies that the derivative of the function $s(1-s)(h(s)-\eta(P_H-P_L)/w)$ at the ESS $s^* = h^{-1}((\eta/w)(P_H-P_L))$ is negative. Hence, the system is asymptotically stable at $s^* = h^{-1}((\eta/w)(P_H-P_L))$. ∎

To study the asymptotic stability of the W-CDMA network under the delayed replicator dynamics (4) around the unique ESS $s^* = h^{-1}((\eta/w)(P_H-P_L))$, we linearize (7) at s^*. Thus, we obtain the following delayed differential equation (DDE):

$$\dot{y}(t) = \mu s^*(1-s^*)\left(y(t-\tau_H)\frac{\partial}{\partial s}f(P_i,s)_{|s=s^*} - y(t-\tau_L)\frac{\partial}{\partial s}f(P_L,s)_{|s=s^*}\right) \tag{8}$$

where τ_H (resp. τ_L) is the time delay of P_H (resp. P_L). The following theorem gives sufficient conditions of stability of (8) at zero.

Theorem 5: Let $P_D = P_H - P_L$. Suppose that at least one of the following conditions holds:

- $\mu\tau_H < ((\Phi_1 + \Phi_2)/(\Phi_1 h^{-1}((\eta/w)P_D)(1 - h^{-1}((\eta/w)P_D))(\Phi_1 + |\Phi_2|)))$;
- $\Phi_1\tau_H + |\Phi_2|\tau_L < ((\Phi_1 + \Phi_2)/(\mu h^{-1}((\eta/w)P_D)(1 - h^{-1}((\eta/w)P_D))(\Phi_1 + |\Phi_2|)))$.

Then, the ESS s^* is asymptotically stable. Moreover, if $\tau_H = \tau_L = \tau$, the ESS is asymptotically stable if and only if

$$\tau < \frac{\pi}{2\mu} \frac{1}{h^{-1}\left(\frac{\eta}{w}P_D\right)\left(1 - h^{-1}\left(\frac{\eta}{w}P_D\right)\right)(|\Phi_1| + |\Phi_2|)}.$$

Proof: To derive the sufficient condition of stability, we need to compute the value of $(\partial/\partial s) f(P_i, s)$ at $s = s^*$. Applying the rule of Lebesgue integration of a function with several parameters, one has

$$\frac{\partial}{\partial s} f(P_i, s)_{|s=s^*} = w \int_0^R \frac{\left[\frac{\partial}{\partial s}\mathrm{SINR}(P_i, s, r)\right]_{s=s^*}}{1 + \mathrm{SINR}(P_i, s^*, r)} \varsigma(r) dr.$$

Define T as $T(r) = 1/r_0^\alpha$ if $r \le r_0$ and as $1/r^\alpha$ if $r \ge r_0$. Since $\Pi := \mathrm{SINR}(P_i, s, r) = ((gP_i/r_0^\alpha)/(\sigma + \beta I(s)))$ if $r \le r_0$ and is equal to $(gP_i/r^\alpha/\sigma + \beta I(s))$ otherwise, where $I(s) = (s(P_H - P_L) + P_L)c(r)$, then

$$c(r) = \begin{cases} \lambda\pi g\left[\frac{\alpha}{\alpha-2} r_0^{-(\alpha-2)} - 2(1-\mu)R^{-(\alpha-2)}\right], & \text{if } r \ge r_0 \\ \frac{\lambda\pi g}{r_0^{\alpha-2}}, & \text{otherwise.} \end{cases}$$

Thus

$$\left.\frac{\partial}{\partial s}\Pi\right|_{s=s^*} = -\frac{gP_i T(r)c(r)\beta(P_H - P_L)}{(\sigma + \beta I(s^*))^2} < 0.$$

Let

$$\Phi_1 := -W \int_0^R \frac{T(r)c(r)\varsigma(r)}{(\sigma + \beta I(s^*))^2 (1 + \mathrm{SINR}(P_H, s^*, r))} dr,$$

$$\Phi_2 := W \int_0^R \frac{T(r)c(r)\varsigma(r)}{(\sigma + \beta I(s^*))^2 (1 + \mathrm{SINR}(P_L, s^*, r))} dr$$

where $W = -wgP_H\beta(P_H - P_L)$. One has $\Phi_1 + \Phi_2 > 0$. A sufficient condition of the stability of the ESS using the stability of the trivial solution of the DDE (8) is then given by $\mu\tau_H < ((\Phi_1 + \Phi_2)/(\Phi_1 h^{-1}((\eta/w)P_D)(1 - h^{-1}((\eta/w)P_D))(\Phi_1 + |\Phi_2|)))$. Note that this stability condition is independent of τ_L. The other results are derived as in Theorem 3. ∎

C. More Than Two Power Levels

For continuously differentiable evolutionary game dynamics, a local stability and asymptotic stability areas of equilibria and the ESS (when it exists) can be established by linearizing the DDEs at the rest point. When an interior rest point exists under the nondelayed replicator, a necessary and sufficient condition of stability under the delayed replicator dynamics is that all roots of the following *characteristic equation*:

$$\det\left(\lambda I - K\sum_{b\in\mathcal{A}} B^b e^{-\tau_b\lambda}\right) = 0 \tag{9}$$

have negative real parts. I is the identity matrix with the same size as the matrix B^b, which is the Jacobian of the system at the rest point. This transcendental equation in λ is, in general, difficult to solve. If x is stable under the nondelayed replicator dynamics, then a sufficient condition of stability under the delayed replicator dynamics is obtained for specific norms of the matrix $K\sum_{b\in\mathcal{A}} B^b$. For example, we can consider $K\sum_{b\in\mathcal{A}} \tau_b\|B^b\|_\infty < 1$, where $\|B^b\|_\infty = \max_{i,j}|B^b_{ij}|$.

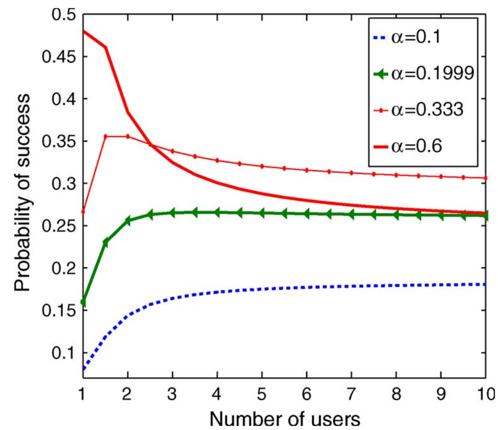

Fig. 6. Impact of n in the probability of success in the Dirac distribution.

VI. Numerical Investigation

A. Slotted Aloha-Based Wireless Networks

We first show the impact of the density of nodes on the probability of success at ESS equilibrium with different values of α, which is the ratio between the collision cost and the global reward (benefit minus global cost) of a user. In all examples, we consider a probability $\mu = 0.8$ for each node of having a receiver in its range.

1) Optimization of the Total Throughput: In Fig. 6, we observe the total throughput $\beta(\alpha, n)$ depending on the number of nodes n and with different values of α. For $\alpha = 1/3$, we observe that the total throughput is increasing in that case with the number of nodes, which seems nonintuitive. The reason is that the number of transmitted mobiles at the ESS, i.e., s^*, is exponentially decreasing with n. Another important result is that it may have a finite number of interferers that maximize the total throughput like in Fig. 6 with $\alpha = 0.2$. When the ratio α is small, the total throughput is decreasing in n, as shown in Fig. 6 for $\alpha = 0.05$. Then, depending on the cost parameters, we have a different behavior of the total throughput in the function of the density of nodes.

In Fig. 7, we represent the probability of success $\beta(\alpha, n)$ as a function of α for several values of n. We observe that the probability of success at optimal value $\alpha^* = (1 - 1/n)^{n-1}$ is increasing with n and tends to the value $1/e$. We observe the

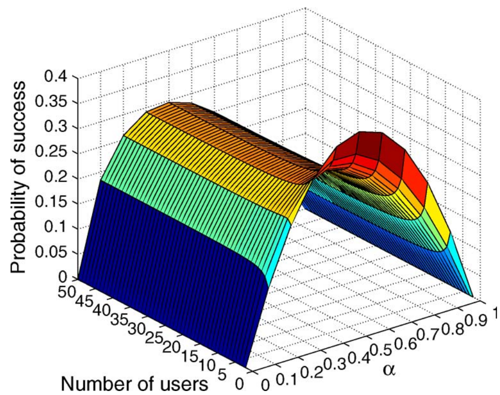

Fig. 7. Probability of success in the Dirac distribution as a function of (α, n).

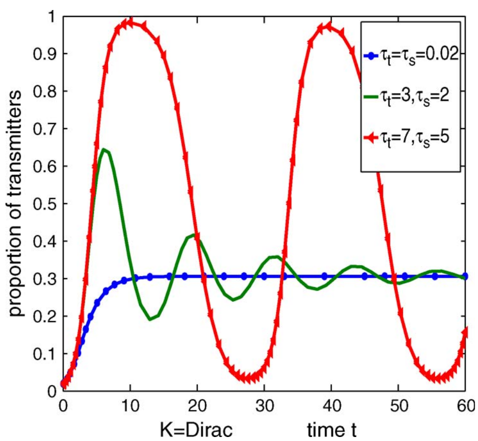

Fig. 8. Delay effect in the Dirac distribution.

same behavior when the nodes are randomly distributed over a plan following a Poisson distribution.

2) Dynamics: Now, we study the effect of the time delays on the convergence of the replicator dynamics to the evolutionary stable strategies in which each pure strategy is associated with its own delay. In Fig. 8, we plot the evolution of the fraction of transmitters for different values of delays when the random variable K is a Dirac $\delta_{\{n-1\}}$. We took $n = 4$, $\Delta = \delta = 1/4$, and $V = 1$. The initial condition is 0.02, and the delays τ_T and τ_S are between 0.02 and 7. For the small delays $(\tau_T, \tau_S) = (0.02, 0.02)$ and $(\tau_T, \tau_S) = (3, 2)$, the system is stable. For the delays $\tau_H = 7$ and $\tau_S = 5$, the system is unstable, and the proportion of transmitters in the cell oscillates around the ESS.

In Fig. 9, we describe the numerical application of our evolutionary game model with a Poisson distribution. We took $k_{\max} = 4$, $\Delta = \delta = \kappa = 1/4$, $\lambda = 1$, and $V = 1$. The initial condition in all these figures is 0.02. In Fig. 9, we compare the evolution of the fraction of transmitters varying the parameter of density λ between 0.1 and 5 for cases 1, 2, and 3, respectively. We observe that we have stability for all cases.

Now, we study the effect of the time delays on the convergence of the replicator dynamics to the ESSs in which each pure strategy is associated with its own delay. The fraction of transmitters in the population is represented in Fig. 10 for $\lambda = 0.5$ and $r = 1$. The delays τ_T and τ_S are between 0.02 and 7. The system is stable for $\tau_T = \tau_S = 0.02$ or $\tau_T = 3$, $\tau_S = 2$. For $\tau_T = 7$ and $\tau_S = 5$, the system is unstable. We

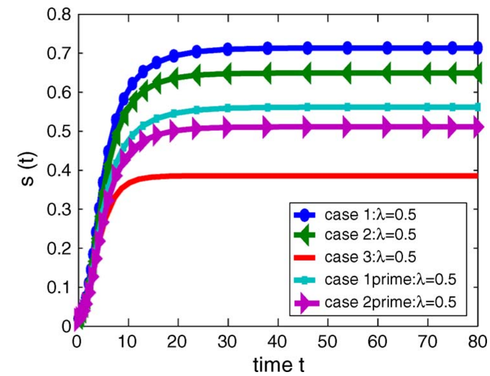

Fig. 9. Evolution of the fraction of transmitters versus λ (without delays).

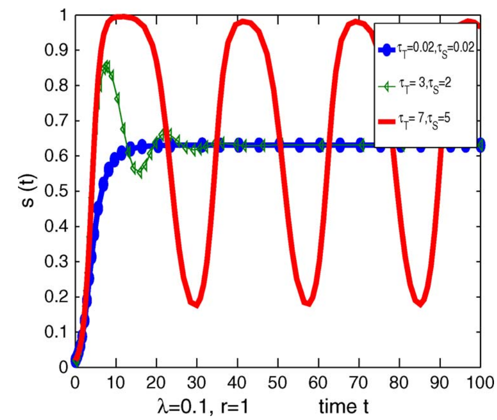

Fig. 10. Impact of the time delay on the stability of the replicator dynamics (case 1).

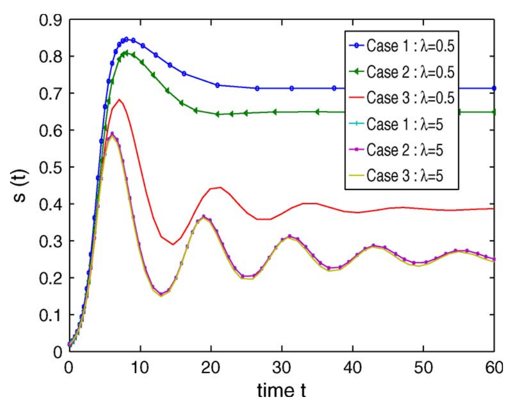

Fig. 11. Evolution of the fraction of transmitters versus λ.

display an oscillatory behavior of the population as a function of time. Trajectories are seen to converge to periodic ones. All turn out to confirm stability conditions that we obtained in Theorem 3. In Fig. 11, we compare the evolution of the fraction of transmitters varying the parameter of density λ between 0.1 and 5 for cases 1, 2, and 3, respectively. In this figure, the time delays are 3 and 2, respectively. Note that, in this figure, the equilibrium point is a decreasing function in the density parameter λ. Indeed, when the density of nodes increases, the number of mobiles that share a receiver increases. To avoid collision, the nodes decrease the probability of transmission. We also observe that for $\lambda = 5$, we have stability, but the convergence speed is slower than for $\lambda = 0.1$.

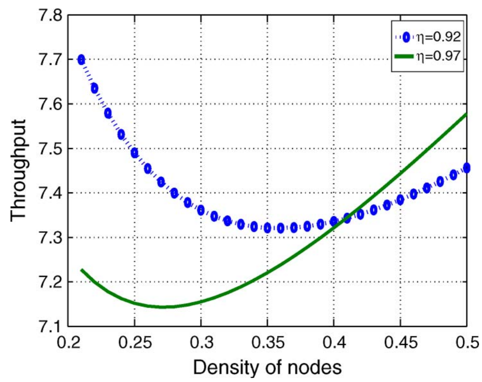

Fig. 12. Average rate of a mobile at equilibrium versus the density of nodes λ for $\eta = 0.92, 0.97$.

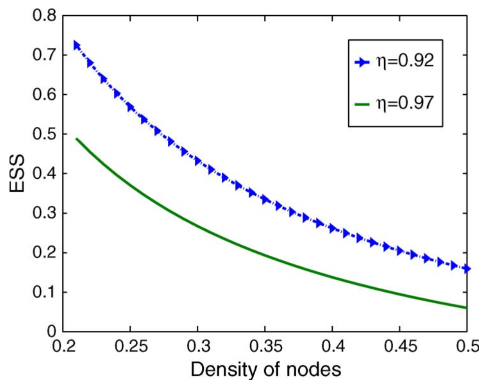

Fig. 13. ESS versus the density of nodes λ for $\eta = 0.92, 0.97$.

B. W-CDMA Wireless Networks

1) Optimization of the Average Throughput: We first show the impact of the density of nodes and pricing on the ESS and the average throughput. We assume base stations that are randomly placed over a plane following a Poisson process with density ν, i.e., $\varsigma(r) = \nu e^{-\nu r}$. We recall that the rate of a mobile using power level P_i at the equilibrium is given by $w\int_0^R \log(1+\mathrm{SINR}(P_i, s, r))\varsigma(r)dr$. We took the following parameters: $r_0 = 0.2$, $w = 20$, $\sigma = 0.2$, $\alpha = 3$, $\beta = 0.2$, and $R = 1$. First, we show the impact of the density of nodes λ on the ESS and the average throughput. In Figs. 12 and 13, we depict the average throughput obtained at the equilibrium and the ESS, respectively, as a function of the density λ. We recall that the interference for a mobile increases when λ increases. We observe that the mobiles become less aggressive when the density increases. In Fig. 12, we observe that it is important to adapt the pricing as a function of the density of nodes. Indeed, we observe that for a low density of nodes, lower pricing $(\eta = 0.92)$ gives better results than higher pricing $(\eta = 0.97)$. When the density of the nodes increases, better performance is obtained with higher pricing.

2) Dynamics: Now, we study the impact of the receiver distributions on the ESS. Fig. 14 represents the fraction of the population using the high power level for different initial states of the population: 0.99, 0.66, 0.25, and 0.03. We observe that the choice of the receiver distributions changes the ESS. For the impact of the time delay on the convergence of the replicator dynamics to the ESS, we obtain the same behavior as in the slotted Aloha-based wireless networks.

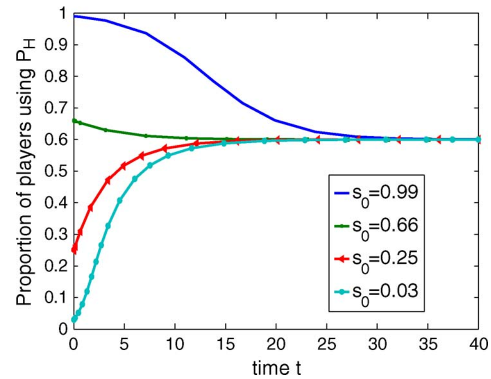

Fig. 14. Convergence to the ESS in the W-CDMA system: quadratic distribution.

VII. Concluding Remarks

This paper has illustrated the potential of evolutionary games to study new robust equilibrium concepts and to describe the dynamics of competition in networking. We have done so through a study of an access game and a power control problem. To be applicable to these problems, we have extended the classical pairwise interaction model of evolutionary games, which has been used in biology to general types of interactions, to be more appropriate for networking.

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