Networks with discrete dynamics

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Project-team TREC

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Networks with discrete dynamics

Dynamics

- Discrete (time or events).
- Markovian.

Results

- Stability.
- Performance evaluation/bounds.
- Control.

Models

- Queueing networks.
- Probabilistic cellular automata.

Main techniques

- Simulation (perfect sampling).
- Markov reward/decision processes.
- Stochastic orders.
- Percolation.

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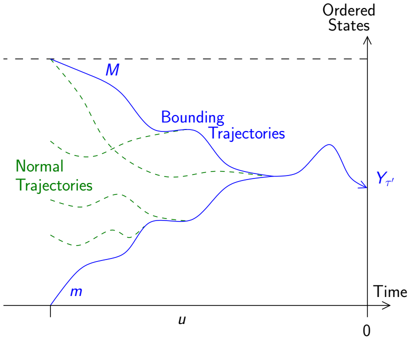
Zoom: Envelope perfect sampling



Challenges:

- Envelope computation.
- Coupling time.
- State space (without lattice structure, infinite case).

Almost Space Homogeneous Events

B., Gaujal, Pin. Performance Evaluation 2012.

$\mathcal{X} = [0, C]$, where $C = (C_1, \ldots, C_d) \in \mathbb{Z}^d$ and $x \leq y \;\Leftrightarrow\; x_i \leq y_i, \; \forall i$;
$[m, M] = [m_1, M_1] \times \cdots \times [m_d, M_d]$.

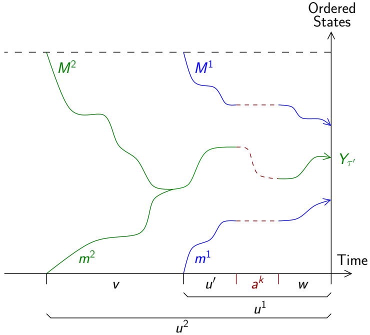
An event a is an almost space homogeneous event (ASHE) if there exists a vector $v \in \mathbb{Z}^n$ (direction vector) and a binary relation $\mathcal{R}$ (blocking relation) on $\{1, \ldots, d\}$ s.t. for all x in $\mathcal{X}$:
Let $B(x) \stackrel{\text{def}}{=} \{i \;:\; x_i + v_i \notin [0, C_i] \text{ and } \exists j \in CR(x), \; (j, i) \in \mathcal{R}\}$ the set of blocked components in x.

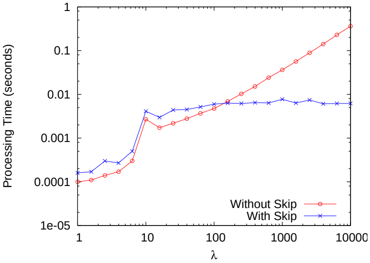
For all i :

$$(x \cdot a)_i = \begin{cases} x_i, & i \in B(x), \\ [(x_i + v_i) \wedge C_i]^+, & i \notin B(x). \end{cases}$$

Complexity of envelope computation: $O(|\mathcal{R}|) = O(d^2)$.

Skipping of events



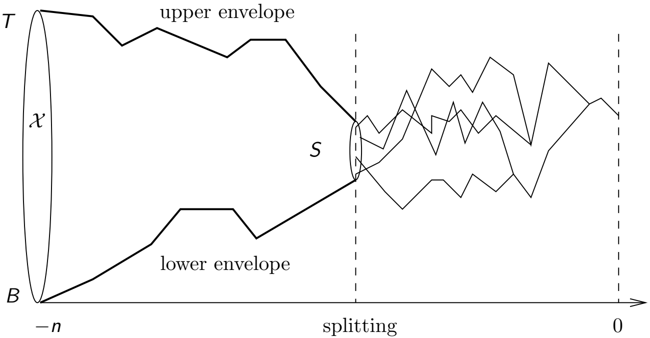


Pin, B., Gaujal. Valuetools 2011.

Beyond envelopes

When the coupling time for envelopes is too long
(or if they do not couple):

- bounds
- splitting



Zoom: Cellular automata

Cell space: any finite or countable group $(G, +)$.
Alphabet: a finite set $\mathcal{A}$.

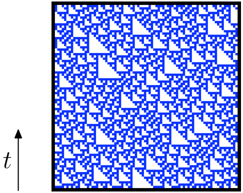
Definition by S. Ulam and J. von Neumann (50's)

A **cellular automaton** is a function $F : \mathcal{A}^G \to \mathcal{A}^G$ characterized by

- a finite neighborhood $V \subset G$,
- a local function $f : \mathcal{A}^V \to \mathcal{A}$ such that $F(x)_k = f((x_{k+v})_{v \in V})$.

Applications

- math. model for distributed computing.
- very simple description generating complex behaviors, models for physical and biological processes.
- CA $\Leftrightarrow$ continuous functions commuting with the shift (Hedlund, 1969)



$G = (\mathbb{Z}, +)$, $\mathcal{A} = \{0, 1\}$,
$V = (0, 1)$,

$F(x)_n = x_n + x_{n+1} \mod 2$.

Probabilistic CA

$\mathcal{M}(\mathcal{A})$ the set of probability measures on $\mathcal{A}$.

A PCA is given by

- a finite neighbourhood $V \subset G$,
- a local function $\varphi : \mathcal{A}^V \to \mathcal{M}(\mathcal{A})$.

The cells are updated synchronously and independently, according to φ (depending on a finite neighbourhood).

This defines an application $F : \mathcal{M}(\mathcal{A}^G) \to \mathcal{M}(\mathcal{A}^G)$, $\mu \mapsto \mu F$.
($\mathcal{M}(\mathcal{A}^G)$ prob. measures on $\mathcal{A}^G$)

Def. A PCA is ergodic if it has a unique invariant measure π, and if for each measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$, the sequence μF^n converges weakly to π (*i.e.* $\mu F^n(C)$ conv. to $\pi(C)$ for any finite cylinder C).

The ergodicity of PCA is undecidable Toom et al., 1990.

Ergodicity of PCA on $\mathbb{Z}$

Set of cells: $\mathbb{Z}$. Even the ergodicity of DCA is undecidable.
B., Mairesse, Marcovici, STACS 2011.

Sufficient conditions: using coupling from the past.

New alphabet $\mathcal{B} = \{0, 1, ?\}$ (unknown letters replaced by "?").
$g : \mathcal{B}^V \to \mathcal{M}(\mathcal{B})$, defined for each $y \in \mathcal{B}^V$ by

$$g(y)(0) = \min_{x \in \mathcal{A}^V,\, x \in y} f(x)(0), \quad g(y)(1) = \min_{x \in \mathcal{A}^V,\, x \in y} f(x)(1),$$

$$g(y)(?) = 1 - \min_{x \in \mathcal{A}^V,\, x \in y} f(x)(0) - \min_{x \in \mathcal{A}^V, x \in y} f(x)(1).$$

Theorem. There exists a critical value $0 < \alpha^* < 1$, depending only on $|V|$, such that F is ergodic if

$$g(?^V)(?) < \alpha^*.$$

Density classification

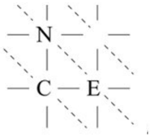
$\mathcal{A} = \{0, 1\}$, $p \in [0, 1]$. Initial distrib. μ_p on $\mathcal{A}^{\mathbb{Z}}$: i.i.d., 1 with prob. p.

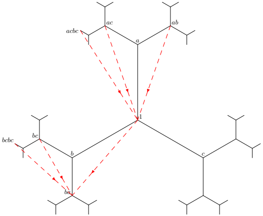
Question: Find a CA (or PCA) such that

$$\begin{cases} p < 1/2 \implies \mu_p F^n \xrightarrow[n\to\infty]{w} \delta_{0^G} \\ p > 1/2 \implies \mu_p F^n \xrightarrow[n\to\infty]{w} \delta_{1^G} \end{cases}.$$

Results:

$\mathbb{Z}^2$: Toom's rule





$\mathbb{Z}$: conjectured candidates

$T_3 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle$,

$F(x)_g = \mathrm{maj}(x_{gab}, x_{gac}, x_{gacbc})$.

B., Fatès, Mairesse, Marcovici, LATIN 2012

Other results

Bounding techniques for Markov chains

Problem: combinatorial explosion of state space

Objective: to find another chain that provides bounds for the original chain and that is simpler to analyze.

Main result: B., Vliegen, Scheller-Wolf, Math. Oper. Res. 2012.

- Upper/lower bounds for steady-state reward functions by redirecting transitions to more/less attractive sets of states.
- Simplifying the model using aggregation of states.
- Application: ATO systems.

Bipartite matching

- Dynamic (queueing) variant.
- Stability (of a max-weight type policy; global)

Projects and collaborations

ANR MAGNUM (2010-14): random generation of complex combinatorial structures and simulation of random dynamical discrete systems.
Partners: UPMC, Paris Diderot, University Paris Nord.

ARC OCOQS (2011-12): Optimal threshold policies in COntrolled Queuing Systems.
With: Hyon (University of Paris Ouest Nanterre), Jean-Marie (INRIA, MAESTRO), Vliegen (University of Twente).

Perfect sampling: Gaujal (INRIA, MESCAL)

Markov chains: Fourneau (UVSQ), Scheller-Wolf (Carnegie Mellon).

Bipartite matching: Gupta (Google), Mairesse (CNRS)

PCA: Mairesse (CNRS), Fatès (INRIA, MAIA).